Saddle Cycles: Solving Rational Expectations Models Featuring Limit Cycles (or Chaos) Using Perturbation Methods

ONLINE APPENDIX

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A General Solution of First-Order Approximation

Without loss of generality, assume henceforth that the re-ordering of a QZ decomposition always places the n_∞ zeros appearing on the main diagonal of S in the lower-right $n_\infty \times n_\infty$ block. Given such a re-ordering, the following proposition summarizes the key results needed to generate a set of first-order approximations.

Proposition A.1. *Suppose (T, S, U, R) is a real QZ decomposition as described in the text, so that the first n_y columns of U are a basis for a $w \in \mathcal{W}^*$. Partition*

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ 0 & T_{22} & T_{23} \\ 0 & 0 & T_{33} \end{pmatrix}, \qquad S = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ 0 & S_{22} & S_{23} \\ 0 & 0 & S_{33} \end{pmatrix},$$

$$R = \begin{pmatrix} R_1 & R_2 & R_3 \end{pmatrix}, \qquad U = \begin{pmatrix} U_{yy} & U_{yz} \\ U_{zy} & U_{zz} \end{pmatrix},$$

where the rows and columns of T and S, and the columns of R, are partitioned into groups of size n_y, $n_z - n_\infty$, and n_∞ (in that order), and the columns and rows of U are partitioned into groups of size n_y and n_z. Then $S_{33} = 0$ whenever it has non-zero dimension (i.e., whenever $n_\infty \geq 1$), and

$$\phi_\theta = U_{zz}^{\top -1}\Psi, \qquad \phi_y = -U_{zz}^{\top -1}U_{yz}^{\top}, \qquad \phi_\zeta = 0,$$

$$\pi_\theta = U_{yz}\Psi B_\theta - U_{yy}\Phi_1 - \pi_y U_{yz}\Psi, \qquad \pi_y = U_{yy}S_{11}^{-1}T_{11}U_{yy}^{-1}, \qquad \pi_\zeta = 0,$$

where $\Phi_1 \equiv -S_{11}^{-1}(T_{yz}\Psi - S_{yz}\Psi B_\theta + R_1^\top C)$, $T_{yz} \equiv (T_{12}, T_{13})$, $S_{yz} \equiv (S_{12}, S_{13})$, *and*

$$\Psi = \begin{pmatrix} \Psi_2 \\ \Psi_3 \end{pmatrix} ,$$

where $\Psi_3 \equiv -T_{33}^{-1}R_3^\top C$, *and* Ψ_2 *is the solution to the Sylvester equation*[1]

$$S_{22}^{-1}T_{22}\Psi_2 - \Psi_2 B_\theta = \Phi_2 \,, \tag{A.1}$$

where, if $n_\infty \geq 1$, $\Phi_2 \equiv -S_{22}^{-1}(T_{23}\Psi_3 - S_{23}\Psi_3 B_\theta + R_2^\top C)$, *while if* $n_\infty = 0$ *we have* $\Phi_2 \equiv -S_{22}^{-1}R_2^\top C$.

B Proofs of Propositions

Proof of Proposition 1

First, note that $w \in \mathcal{W}$ if and only if (a) w is a linear subspace, and (b) $x \in w$ implies $Ax \in w$ (i.e., w is A-invariant). Suppose (π, ϕ) satisfies (2). We verify that $W(\phi) \in \mathcal{W}$. $W(\phi)$ is clearly a linear subspace, so it remains only to verify that it is A-invariant. Now, (π_y, ϕ_y) must satisfy (4), which can be re-written

$$\Gamma_x \begin{pmatrix} I_{n_y} \\ \phi_y \end{pmatrix} + \Gamma_{x'} \begin{pmatrix} I_{n_y} \\ \phi_y \end{pmatrix} \pi_y = 0 \,.$$

Pre-multiplying by $-\Gamma_{x'}^{-1}$, using the definition of A, and re-arranging, we get

$$A \begin{pmatrix} I_{n_y} \\ \phi_y \end{pmatrix} = \begin{pmatrix} I_{n_y} \\ \phi_y \end{pmatrix} \pi_y \,. \tag{A.2}$$

Next, $x \in W(\phi)$ if and only if $z = \phi_y y$ for some $y \in \mathbb{R}^{n_y}$. Thus, take $x \in W(\phi)$, so that we may write

$$x = \begin{pmatrix} I_{n_y} \\ \phi_y \end{pmatrix} y \,.$$

[1] While Sylvester equations can be analytically solved using vectorization and the Kronecker product, it is typically much faster to solve them using efficient numerical routines. This can be done for Ψ_2 using the function `sylvester` in MATLAB and Julia, the SLICOT routine `SB04MD` in Fortran, or the Armadillo routine `syl` in C++. Note that a solution for Ψ_2 generally only exists when $S_{22}^{-1}T_{22}$ and B_θ do not share any eigenvalues. It can be verified that the eigenvalues of $S_{22}^{-1}T_{22}$ are precisely the generalized eigenvalues of (A, B) that are *not* associated with the RGEs making up w. Thus, if λ is both an eigenvalue of B_θ and a generalized eigenvalue of (A, B), then the corresponding RGEs would always need to be included in w.

We need to verify that $x^* \equiv Ax \in W(\phi)$. We have

$$x^* = A\begin{pmatrix} I_{n_y} \\ \phi_y \end{pmatrix} y = \begin{pmatrix} I_{n_y} \\ \phi_y \end{pmatrix} \pi_y y = \begin{pmatrix} I_{n_y} \\ \phi_y \end{pmatrix} y^*\,,$$

where the second equality uses (A.2) and $y^* \equiv \pi_y y$. Thus $z^* = \phi_y y^* \in W(\phi)$, which confirms that $W(\phi)$ is A-invariant. □

Proof of Proposition 2

Suppose w is as stated in the Proposition. Let $A = UTU^\top$ be a real Schur decomposition[2] ordered so that the n_y eigenvalues associated with w are the eigenvalues of the upper-left $n_y \times n_y$ diagonal block of T, and the first n_y columns of U are a basis for w. Note that U is an orthogonal matrix ($UU^\top = I_n$) and T is block upper-triangular. Partition U and T conformably with $x = (y, z)$ as

$$U = \begin{pmatrix} U_{yy} & U_{yz} \\ U_{zy} & U_{zz} \end{pmatrix} \quad , \quad T = \begin{pmatrix} T_{yy} & T_{yz} \\ 0 & T_{zz} \end{pmatrix},$$

and for any x let $\tilde{x} = (\tilde{y}, \tilde{z}) \equiv U^{-1}x = U^\top x$. Since the first n_y columns of U are a basis for w, for any $x \in w$ we must have $\tilde{z} = 0$, i.e., we must have $U_{yz}^\top y + U_{zz}^\top z = 0$, which can be solved to obtain $z = \psi y$, where

$$\psi = -U_{zz}^{\top -1} U_{yz}^\top\,.$$

Further, since by definition $y = U_{yy}\tilde{y} + U_{yz}\tilde{z}$, if $x \in w$ then $\tilde{z} = 0$ and thus $y = U_{yy}\tilde{y}$, which can be solved to obtain $\tilde{y} = U_{yy}^{-1} y$.

Next, take $x \in w$, and let $x^* = Ax$, which is by construction also an element of w. Pre-multiplying both sides by $U^\top$ and using the Schur decomposition of A, we have $\tilde{x}^* = T\tilde{x}$. Since $\tilde{z} = 0$, it follows that $\tilde{y}^* = T_{yy}\tilde{y}$. Substituting in $\tilde{y} = U_{yy}^{-1} y$ and $\tilde{y}^* = U_{yy}^{-1} y^*$, then pre-multiplying both sides by U_{yy}, we obtain that $y^* = \kappa y$, where

$$\kappa \equiv U_{yy} T_{yy} U_{yy}^{-1}$$

Clearly, the eigenvalues of κ are the eigenvalues of T_{yy},[3] which are in turn the eigenvalues associated with the RGEs making up w.

[2] See the discussion in Section 3.2.

[3] $\kappa v = \lambda v$ if and only if $T_{yy} U_{yy}^{-1} v = \lambda U_{yy}^{-1} v$, and therefore (λ, v) is an eigenvalue-eigenvector pair of κ if and only if $(\lambda, U_{yy}^{-1} v)$ is an eigenvalue-eigenvector pair of T_{yy}.

We now show that $(\pi_y, \phi_y) = (\kappa, \psi)$ is a solution to (4). As shown in the proof of Proposition 1, this is equivalent to showing that (A.2) holds for this choice, which in turn holds if and only if

$$A \begin{pmatrix} I_{n_y} \\ \psi \end{pmatrix} y = \begin{pmatrix} I_{n_y} \\ \psi \end{pmatrix} \kappa y \tag{A.3}$$

holds for all $y \in \mathbb{R}^{n_y}$. Choose any $y \in \mathbb{R}^{n_y}$, and set $z = \psi y$, so that

$$x = \begin{pmatrix} I_{n_y} \\ \psi \end{pmatrix} y \in w\,.$$

Since w is A-invariant, we have $x^* \equiv Ax \in w$, and in particular, as noted above, $y^* = \kappa y$ and, by construction of w, $z^* = \psi y^*$. Thus

$$A \begin{pmatrix} I_{n_y} \\ \psi \end{pmatrix} y = Ax = \begin{pmatrix} I_{n_y} \\ \psi \end{pmatrix} y^* = \begin{pmatrix} I_{n_y} \\ \psi \end{pmatrix} \kappa y\,.$$

Since y was chosen arbitrarily, (A.3) indeed holds for all $y \in \mathbb{R}^{n_y}$. □

Proof of Proposition 3

To avoid cluttering the notation, we prove the Proposition for the case of Q empty. It is straightforward to extend the proof to any arbitrary Q.

Note first that (2), (3), and Propositions 1-2 do not rely in any way on the distinction between pre-determined and jump variables. Thus, their content would hold for any arbitrary re-ordering and partition of the vector x into two subvectors y and z. In particular, each such re-ordering/partition is associated with a set $\mathcal{S}_\emptyset$ of solutions (π, ϕ) to (2) and a set $\mathcal{W}^*_\emptyset$ given as in Proposition 2. The following lemma will be useful.

Lemma A.1. *Let (y, z) and $(\tilde{y}, \tilde{z})$ be two partitions of x.*[4] *Let $\mathcal{S}_\emptyset$ and $\mathcal{W}^*_\emptyset$ denote the sets associated with (y, z), and $\widetilde{\mathcal{S}}_\emptyset$ and $\widetilde{\mathcal{W}}^*_\emptyset$ the ones associated with $(\tilde{y}, \tilde{z})$. Suppose $w \in \mathcal{W}^*_\emptyset$ and $\tilde{w} \in \widetilde{\mathcal{W}}^*_\emptyset$ are such that $\tilde{w} \subset w$, and let $(\pi, \phi) \in \mathcal{S}_\emptyset$ and $(\tilde{\pi}, \tilde{\phi}) \in \widetilde{\mathcal{S}}_\emptyset$ be the associated solutions to (2). Then $M(\tilde{\phi}) \subset M(\phi)$.*

Proof. Let p denote the dimension of y and q of $\tilde{y}$. It will be useful to re-cast (2) as follows. For a given x, let $h(x)$ be the value of x' solving (1) (which exists uniquely

[4] It is straightforward to extend the Lemma to the case of arbitrary re-orderings of x as well, but doing so requires carrying around the mapping relating the elements of two different re-orderings throughout both the statement of the Lemma and its proof. As this is quite tedious, we omit it here for simplicity.

by Assumption 1), so that x evolves according to $x' = h(x)$. Partitioning $h(x) = (f(x), g(x))$ conformably with $x = (y, z)$, for this partition we may re-write condition (2) as requiring that

$$\begin{pmatrix} \pi(y) \\ \phi(\pi(y)) \end{pmatrix} = \begin{pmatrix} f(y, \phi(y)) \\ g(y, \phi(y)) \end{pmatrix}, \quad \forall y \in \Theta. \tag{A.4}$$

Note that, for such a ϕ, $M(\phi)$ is, by construction, a p-dimensional h-invariant manifold in $\mathbb{R}^n$: if $x \in M(\phi)$ (i.e., $x = (y, \phi(y))$ for some y), then letting $y' = \pi(y) \equiv f(y, \phi(y))$, we have that $h(x) = (y', \phi(y')) \in M(\phi)$. Similarly, partitioning $h(x) = (\tilde{f}(x), \tilde{g}(x))$ conformably with $x = (\tilde{y}, \tilde{z})$, for this partition (2) can be written

$$\begin{pmatrix} \tilde{\pi}(\tilde{y}) \\ \tilde{\phi}(\tilde{\pi}(\tilde{y})) \end{pmatrix} = \begin{pmatrix} f\left(\tilde{y}, \tilde{\phi}(\tilde{y})\right) \\ g\left(\tilde{y}, \tilde{\phi}(\tilde{y})\right) \end{pmatrix}, \quad \forall \tilde{y} \in \widetilde{\Theta},. \tag{A.5}$$

and $M(\tilde{\phi})$ is also h-invariant.

Let $V : \mathbb{R}^p \to M(\phi)$ be a smooth invertible function whose inverse is also smooth. Such functions necessarily exist since $M(\phi)$ is a p-dimensional manifold. Thus, $X \equiv V^{-1}(x)$ is a coordinate system on $M(\phi)$, where the change between coordinates $X \in \mathbb{R}^p$ and $x \in M(\phi)$ is smooth. Since $M(\phi)$ is an h-invariant manifold, we can therefore write the restriction of the system $x' = h(x)$ to $M(\phi)$ as

$$X' = H(X) \equiv V^{-1}(h(V(X))),$$

with $x = V(X)$.

Take $\tilde{w}, w$ as in the statement of the Proposition. Since $\tilde{w} \subset w$, $q \leq p$. To make the problem non-trivial, assume $q < p$. Let $A = UTU^\top$ be a real Schur decomposition such that the first q columns of U are a basis for $\tilde{w}$ and the first p columns a basis for w (which is possible since we've taken $\tilde{w} \subset w$). Without loss of generality, choose V such that V_X is equal to the first p columns of U, and note that this implies $V_X^\top V_X = I_p$. Then the linearized system $x' = Ax$ restricted to $W(\phi)$ can be written

$$X' = \widehat{A}X,$$

with $x = V_X X$, where $\widehat{A} \equiv V_X^\top A V_X = T_{yy}$, and T_{yy} is the upper-left $p \times p$ block of T. $\widehat{A}$ thus has the eigenvalues associated with the RGEs that make up w (see proof of Proposition 2). Let Ω be the set of non-trivial $\widehat{A}$-invariant linear subspaces of $\mathbb{R}^p$.

Partition $X = (Y, Z)$, where Y is a q-vector, and partition $H = (F, G)$ conformably. Let (Π, Φ) be a pair of functions satisfying (A.4) for the restricted system, i.e.,

$$\begin{pmatrix} \Pi\left(Y\right) \\ \Phi\left(\Pi\left(Y\right)\right) \end{pmatrix} = \begin{pmatrix} F\left(Y, \Phi\left(Y\right)\right) \\ G\left(Y, \Phi\left(Y\right)\right) \end{pmatrix}, \quad \forall y \in \mathbb{R}^q\,, \tag{A.6}$$

and denote the set of such pairs Σ. Analogous to the results presented in Section 2.2, elements of Σ are indexed by elements of Ω^*, where Ω^* is the subset of elements ω of Ω that can be written

$$\omega = \{(Y, Z) \in \mathbb{R}^p : Y \in \mathbb{R}^q,\ Z = \psi Y\}$$

for some matrix ψ.

Next, note that, since $\widehat{A} = T_{yy}$, where T_{yy} is the upper-left $p \times p$ block of T, we see that $\widehat{A} = \widehat{U} T_{yy} \widehat{U}^\top$ with $\widehat{U} = I_p$ is a real Schur decomposition of $\widehat{A}$. Let $\tilde{\omega}$ be the space spanned by the first q columns of $\widehat{U}$, i.e., $\tilde{\omega} = \{(Y, Z) \in \mathbb{R}^p : Z = 0\}$. Let (Π, Φ) be the associated element of Σ, and $\mu(\Phi)$ the q-dimensional H-invariant manifold mapped out by $Z = \Phi(Y)$. By construction, the set $\widetilde{M} \equiv V(\mu(\Phi))$ is contained in $M(\phi)$. We now verify that $\widetilde{M} = M(\tilde{\phi})$, which would establish that $M(\tilde{\phi}) \subset M(\phi)$. To see this, partition $V = (V^{\tilde{y}}, V^{\tilde{z}})$ and $V^{-1} = (V^{-Y}, V^{-Z})$ conformably with, respectively, $x = (\tilde{y}, \tilde{z})$ and $X = (Y, Z)$, and note that condition (A.6) is equivalent to

$$V\left(\Pi\left(Y\right), \Phi\left(\Pi\left(Y\right)\right)\right) = h\left(V\left(Y, \Phi\left(Y\right)\right)\right), \quad \forall y \in \Theta\,. \tag{A.7}$$

Now, let

$$K\left(Y\right) \equiv V^{\tilde{y}}\left(Y, \Phi\left(Y\right)\right),$$

and note that K is a bijection from $\mathbb{R}^q$ to $\Theta^* \equiv \{\tilde{y} \in \mathbb{R}^q : (\tilde{y}, \ddot{y}) \in \Theta \text{ for some } \ddot{y}\}$, and in particular, for $x = (\tilde{y}, \tilde{z}) \in \widetilde{M}$, $\tilde{y} = K(V^{-Y}(x))$. Define also the functions

$$\begin{aligned} \tilde{\pi}^*\left(\tilde{y}\right) &\equiv K\left(\Pi\left(K^{-1}\left(\tilde{y}\right)\right)\right), \\ \tilde{\phi}^*\left(\tilde{y}\right) &\equiv V^{\tilde{z}}\left(K^{-1}\left(\tilde{y}\right), \Phi\left(K^{-1}\left(\tilde{y}\right)\right)\right). \end{aligned}$$

Then condition (A.7) can be written

$$\begin{pmatrix} \tilde{\pi}^*\left(\tilde{y}\right) \\ \tilde{\phi}^*\left(\tilde{\pi}^*\left(\tilde{y}\right)\right) \end{pmatrix} = \begin{pmatrix} \tilde{f}\left(\tilde{y}, \tilde{\phi}^*\left(\tilde{y}\right)\right) \\ \tilde{g}\left(\tilde{y}, \tilde{\phi}^*\left(\tilde{y}\right)\right) \end{pmatrix}, \quad \forall \tilde{y} \in \Theta^*\,. \tag{A.8}$$

Comparing this with (A.5), we see that $(\tilde{\pi}^*, \tilde{\phi}^*) \in \widetilde{S}$ and $\widetilde{M} = M(\tilde{\phi}^*)$. Using the same reasoning as in Section 2.2, solutions to (A.8) are determined uniquely by the element $\tilde{w}^* \in \widetilde{W}^*$ that is tangent to $M(\tilde{\phi}^*)$. To confirm that $\widetilde{M} = M(\tilde{\phi})$, we need therefore only verify that $\tilde{w}^* = \tilde{w}$. Since (i) $\widetilde{M} \equiv V(\mu(\Phi))$, (ii) $\tilde{\omega}$ is tangent to $\mu(\Phi)$, and (iii) $\tilde{w}^*$ is tangent to $\widetilde{M}$, we must have $\tilde{w}^* = V_X\tilde{\omega}$. Now, $\tilde{\omega}$ is the set of p-vectors with only the first q elements non-zero. Thus, $V_X\tilde{\omega}$ is equal to the span of the first q columns of V_X. But V_X is the first p columns of U, and by construction, its first q columns are a basis for $\tilde{w}$. Thus, $\tilde{w}^* = V_X\tilde{\omega} = \tilde{w}$, and the proof is complete.

□

Corollary A.1. *Let (π, ϕ) and $(\tilde{\pi}, \tilde{\phi})$ be as in Lemma A.1. Then for any $x_0 \in M(\tilde{\phi})$, the sequence $x_t = (y_t, z_t)$ given by $y_t = \pi^t(y_0)$, $z_t = \phi(y_t)$, and the sequence $\tilde{x}_t = (\tilde{y}_t, \tilde{z}_t)$ given by $\tilde{y}_t = \tilde{\pi}^t(y_0)$, $z_t = \tilde{\phi}(y_t)$ are identical, i.e., $x_t = \tilde{x}_t$ for all t.*

Proof. By construction, if $x \in M(\phi)$, then $x' = h(x)$ implies $(y', z') = (\pi(y), \phi(\pi(y)))$. Thus, for any $x_0 \in M(\phi)$, the sequence generated by $x_{t+1} = h(x_t)$ is precisely the sequence generated by $y_t = \pi^t(y_0)$, $z_t = \phi(y_t)$. Similarly, for any $x_0 \in M(\tilde{\phi})$, the sequence generated by $x_{t+1} = h(x_t)$ is precisely the one generated by $\tilde{y}_t = \tilde{\pi}^t(y_0)$, $z_t = \tilde{\phi}(y_t)$. Since $M(\tilde{\phi}) \subset M(\phi)$ by Lemma A.1, it follows immediately that for any $x_0 \in M(\tilde{\phi})$ (so that also $x_0 \in M(\phi)$), the two sequences in the statement of the Proposition are identical.

□

Consider now part (i) of the Proposition 3. Suppose by hypothesis that λ is real, with $\lambda > 1$. Fix a $(\pi, \phi) \in \mathcal{S}_\emptyset$ such that $r(\lambda) \subset W(\phi)$. Let $x = (\tilde{y}, \tilde{z})$ be a partition where $\tilde{y}$ has only one element, and $\widetilde{\mathcal{S}}_\emptyset$ and $\widetilde{\mathcal{W}}^*_\emptyset$ be as in the statement of Lemma A.1. Let $\tilde{w} \in \widetilde{\mathcal{W}}^*_\emptyset$ be a one-dimensional subset of $r(\lambda)$ (e.g., the space spanned by one of the ordinary eigenvectors associated with λ), and $(\tilde{\pi}, \tilde{\phi})$ the associated element of $\widetilde{\mathcal{S}}_\emptyset$. Since $\tilde{w} \subset r(\lambda) \subset W(\phi)$, we may apply Corollary A.1 to conclude that, if (3) is violated for $(\tilde{\pi}, \tilde{\phi})$, then it is also violated for (π, ϕ): trajectories beginning on $M(\tilde{\phi}) \subset M(\phi)$ do not remain bounded, regardless of how close they begin to zero. Reasoning similar to the discussion immediately following Proposition 3, we see that the one-dimensional graph of $\tilde{y}' = \tilde{\pi}(\tilde{y})$ either crosses the 45-degree more than once, or else it violates (3). Since the former is ruled out by Assumption 1, only the latter is possible, and thus (3) is also violated for (π, ϕ), which verifies part (i).

To show part (ii), suppose to the contrary that we have a unique model solution (π, ϕ), but there exists a λ with $|\lambda| < 1$ such that $w \equiv r(\lambda) \not\subset W(\phi)$. Note that, since the model solution is assumed to be unique, the partition of x associated with (π, ϕ)

is just (y, z). Next, let q be the dimension of w, and re-partition $x = (\tilde{y}, \tilde{z})$, where $\tilde{y}$ has dimension q. By Proposition 2, there exists a solution $(\tilde{\pi}, \tilde{\phi})$ to (2), where $\tilde{\pi}$ and $\tilde{\phi}$ map $\tilde{y}$ into $\tilde{y}'$ and $\tilde{z}$, respectively, with $M(\tilde{\phi})$ tangent to w and such that the q eigenvalues of $\tilde{\pi}_{\tilde{y}}$ are all equal to λ. Since, for $\|\tilde{y}\|$ small enough, the system $\tilde{y}' = \tilde{\pi}(\tilde{y})$ is dominated by its linear part, which converges by hypothesis, this implies that there is an open set containing zero for which $(\tilde{\pi}, \tilde{\phi})$ (which evolves on $M(\tilde{\phi})$) satisfies (3).

If $q \geq n_y$, then $(\tilde{\pi}, \tilde{\phi})$ itself forms part of a second solution, contradicting the supposition that the solution is unique, in which case we are done. Thus, suppose instead $q < n_y$. Let us construct a new model solution $(Q^*, \pi^*, \phi^*, \chi^*)$ as follows. Let $Q^* = \{1, \ldots, n_z\}$, so that (in the notation of Definition 2) $\hat{y}^* = x$, $u^* = z$, and z^* is the empty vector. We then define ϕ^* as the function that always returns the empty vector, and define

$$\pi^*\left(\hat{y}^*\right) \equiv \begin{cases} \left(\tilde{\pi}\left(\tilde{y}\right), \tilde{\phi}\left(\tilde{\pi}\left(\tilde{y}\right)\right)\right) & \text{, if } \hat{y}^* \in M\left(\tilde{\phi}\right) \\ \left(\pi\left(y\right), \phi\left(\pi\left(y\right)\right)\right) & \text{, otherwise} \end{cases} .$$

Finally, let

$$\chi^*\left(y\right) \equiv \begin{cases} \tilde{\phi}^{(z)}\left(\tilde{y}\right) & \text{, if } \left(y, \tilde{\phi}^{(z)}\left(\tilde{y}\right)\right) \in M\left(\tilde{\phi}\right) \\ \phi\left(y\right) & \text{, otherwise} \end{cases} ,$$

where $\tilde{\phi}^{(z)}$ denotes the last n_z elements of $\tilde{\phi}$. It can then be easily verified that $(Q^*, \pi^*, \phi^*, \chi^*)$ is an additional solution, which contradicts the initial supposition that the solution was unique. This completes the proof. □

Proof of Proposition 4

Suppose w, w_1, w_2 are as in the statement of the Proposition. Since $w_j \subset w$, by Lemma A.1 (with appropriate modifications for any re-orderings) we have $M(\phi^j) \subset M(\phi)$. Thus, π^j is just the restriction of π to $M(\phi^j)$. Since (2) is satisfied for π on $x \in M(\phi)$, it must therefore also be satisfied for π^j on $x \in M(\phi^j)$, i.e., (3) must hold, and therefore for any χ^j (e.g., $\chi^j(y) = 0$), $(Q_j, \pi^j, \phi^j, \chi^j)$ is a model solution. Further, since w_1, w_2 are distinct by construction, so are $(Q_1, \pi^1, \phi^1), (Q_2, \pi^2, \phi^2)$. □

Proof of Proposition A.1

To see that $S_{33} = 0$ whenever $n_\infty \geq 1$, suppose instead that $S_{33} \neq 0$. This implies that there is at least one non-zero entry in the bottom-right $n_\infty \times n_\infty$ block of S.

Since the entries in the bottom-left $n_\infty \times (n - n_\infty)$ block of S are necessarily all zero, this implies that the last n_∞ columns of S are not contained in the span of the first $n - n_\infty$ columns, and therefore $\text{rank}(S) > n - n_\infty$. But as noted in the text, $\text{rank}(S) = \text{rank}(B) = n - n_\infty$, which is a contradiction. We therefore conclude that $S_{33} = 0$.

Next, letting $\tilde{x} \equiv U^{-1}x = U^\top x$ and partitioning $\tilde{x} = (\tilde{y}, \tilde{z})$ as $x = (y, z)$, we seek a solution such that $\tilde{z} = \Psi\theta$ for some matrix Ψ. Replacing $A = RTU^\top$ and $B = RSU^\top$ in (16), using the definition of $\tilde{x}$, the partitions given in the statement of the proposition, and substituting in $\tilde{z} = \Psi\theta$ and $\mathbb{E}[\tilde{z}] = \Psi\Pi_\theta\theta$, we obtain

$$\begin{pmatrix} S_{11} & S_{12} & S_{13} \\ 0 & S_{22} & S_{23} \\ 0 & 0 & S_{33} \end{pmatrix} \begin{pmatrix} \mathbb{E}\tilde{y}' \\ \Psi\Pi_\theta\theta \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ 0 & T_{22} & T_{23} \\ 0 & 0 & T_{33} \end{pmatrix} \begin{pmatrix} \tilde{y} \\ \Psi\theta \end{pmatrix} + R^\top C\theta\,. \tag{A.9}$$

Note that, by construction, the eigenvalues of S_{33} are zero, and therefore those of T_{33} must all be non-zero (see **?**), so that T_{33} is invertible. The last n_∞ equations of (A.9) can therefore be written as

$$\Psi_3\theta = -T_{33}^{-1}R_3^\top C\theta\,.$$

Since this must hold for every θ, we obtain $\Psi_3 = -T_{33}^{-1}R_3^\top C$. Given this value, the middle $n_z - n_\infty$ equations in (A.9) can be written

$$\left(S_{22}^{-1}T_{22}\Psi_2 - \Psi_2\Pi_\theta\right)\theta = \Phi_2\theta\,,$$

where Φ_2 is defined in the statement of the proposition. Again, this must hold for every θ, which yields (A.1).

Given the solutions for Ψ_2 and Ψ_3, substituting the definition $\tilde{z} = U_{yz}^\top y + U_{zz}^\top z$ into $\tilde{z} = \Psi\theta$, we can then solve for $z = \phi_\theta\theta + \phi_y y$, where ϕ_θ and ϕ_y are as in the statement of the proposition.

Next, the first n_y equations of (A.9) can be re-arranged as

$$\mathbb{E}\tilde{y}' = S_{11}^{-1}T_{11}\tilde{y} - \Phi_1\theta\,, \tag{A.10}$$

where Φ_1 is as in the statement of the proposition. Note also that $y = U_{yy}\tilde{y} + U_{yz}\tilde{z}$. Substituting in $\tilde{z} = \Psi\theta$, we can solve for $\tilde{y} = U_{yy}^{-1}y - U_{yy}^{-1}U_{yz}\Psi\theta$. Using this to replace $\tilde{y}$ and $\tilde{y}'$ in (A.10), along with $\mathbb{E}[\theta'] = \Pi_\theta\theta$, we obtain $\mathbb{E}[y'] = \pi_\theta\theta + \pi_y y$, where π_θ and π_y are as stated in the proposition.

Finally, as is well known (see, e.g., **?**), to a first-order approximation the system exhibits certainty equivalence, so that we must have $\pi_\zeta = 0$ and $\phi_\zeta = 0$ (though in general higher-order derivatives involving ζ will be non-zero). $\square$

C Perturbation Solution

To improve computational efficiency, we first define $\Omega_t \equiv e_t^{\varphi_e}\lambda_t$ and $\iota_t \equiv Y_{t+1} + \frac{1-\delta-\gamma}{1-\delta}X_t - \frac{1-\delta-\psi}{1-\delta}\gamma Y_t$. This allows us to write system (17) equivalently as

$$\mu_t \lambda_t = Q\left(e_t\right)\mathbb{E}_t\left[\Omega_{t+1}\right] ,$$

$$\lambda_t = \iota_t^{-\omega} ,$$

$$X_{t+1} = \left(1-\delta\right)X_t + \psi Y_{t+1} ,$$

$$Y_{t+1} = z_t e_t^{\alpha} ,$$

$$\Omega_t = e_t^{\varphi_e}\lambda_t ,$$

$$\iota_t = Y_{t+1} + \frac{1-\delta-\gamma}{1-\delta}X_t - \frac{1-\delta-\psi}{1-\delta}\gamma Y_t .$$

This gives us six equations in the two endogenous pre-determined variables (X_t, Y_t) and four endogenous jump variables $(e_t, \lambda_t, \Omega_t, \iota_t)$. We then make the following change of variables. In order to bound the employment rate between 0 and 1, we define $e^* \equiv -\log(1/e-1) \in (-\infty, \infty)$, and then replace all appearances of e in the above system with $(\exp\{-e^*\}+1)^{-1} \in (0,1)$. All other variables (endogenous and exogenous) are log-transformed, so that, for example, we replace X with $\exp\{\widetilde{X}\}$, Y with $\exp\{\widetilde{Y}\}$, etc., where variables with tildes denote logs.

The first-order approximations to elements of $\mathcal{S}$ were found as described in the text. In doing so, Proposition 3(i) was used to rule out RGEs associated with real, positive, unstable eigenvalues. In practice, this always left only a single element of $\mathcal{S}$ as a candidate solution. The second- and third-order approximations were then solved for sequentially using MATLAB functions that were generated automatically using the MATLAB Symbolic Toolbox. To check that this candidate solution satisfied (15), we simulated 1,000 periods of data beginning from an arbitrary initial condition near the non-stochastic SS and verified that the implied employment rate never left the interval $[0.2, 0.9999]$. If it did, we concluded that no solution exists, and the parameterization was discarded.

D Parameter Estimates and Model Fit

Parameter estimates are shown in Table A.I. The fit of the estimated model spectrum is shown in Figure A.1. In each panel, the thick solid curve reports the spectrum obtained from the data,[5] while the thinner gray curve shows the spectrum from the model. Panel (a) shows the results for the discount factor shock, while panel (b) shows the results for the technology shock. In both cases the model fits the spectrum reasonably well. Both models also fit the mean unemployment rate almost exactly, which was 0.0583 in the data sample, 0.0584 in the μ-shock model, and 0.0583 in the z-shock model. The skewness of hours was -0.0922 in the data, which was matched closely in the μ-shock model at -0.0873. The z-shock model fit less well, however, with a skewness of -0.3808.

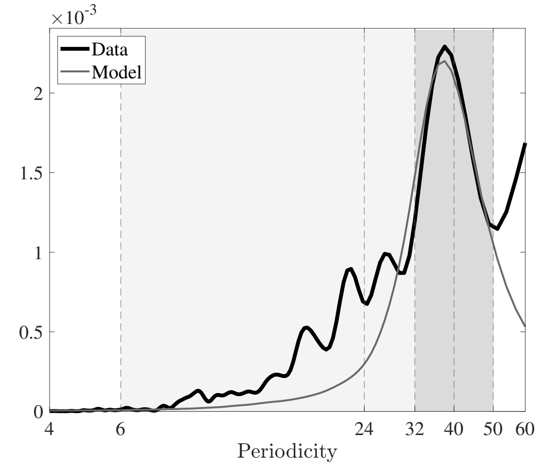
Table A.I: Estimated Parameter Values

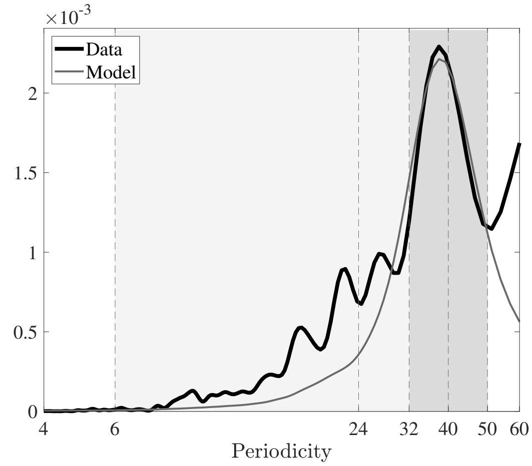
	(a) μ-shock	(b) z-shock
$\bar{e}$	0.9430	0.9455
ω	0.2736	0.2596
γ	0.6259	0.6489
ψ	0.3905	0.3929
φ_e	0.0460	0.0454
ϕ	0.9108	0.9871
$\bar{\Phi}$	0.0470	0.0430
$\bar{\Phi}_2$	0.0018	0.0030
$\bar{\Phi}_3$	0.00066	0.00126
ρ_μ	0.0671	—
σ_μ	0.00014	—
ρ_z	—	0.6254
σ_z	—	0.0027

[5]Note that the horizontal axis in the figures is periodicity (i.e., the length of the underlying cycle, in quarters).

Figure A.1: Estimation Fit

(a) μ-Shock Model (b) z-Shock Model





Notes: Thick black curve shows data spectrum. Thin gray curve shows model spectrum. Data spectrum obtained by computing raw periodogram at 1024 frequencies, then kernel-smoothing the result using a Hamming kernel of length 13. Model spectrum obtained by simulating 500 data sets of same length as actual data set (270 quarters), computing average of the raw periodograms, and then smoothing the result as with the actual data.

E Finite Elements Solution

To implement the FE method for the μ-shock model, in order to bound the shock process, as in Aruoba et al. (2006) we first make the change of variables $M_t \equiv \tanh(\log(\mu_t))$ and $\nu_{\mu,t} \equiv \varepsilon_{\mu,t}/\sqrt{2}$. Letting $s = (X, Y, M)$ be the new state vector, we construct grids $\mathbf{X} \equiv \{X_0, \ldots, X_{n_X+1}\}$, $\mathbf{Y} \equiv \{Y_0, \ldots, Y_{n_Y+1}\}$, $\mathbf{M} \equiv \{M_0, \ldots, M_{n_M+1}\}$ for the state variables. We then construct the ijk-th element as

$$\Psi_{ijk}(s) \equiv \Psi_{X,i}(X)\Psi_{Y,j}(Y)\Psi_{M,k}(M)\,,$$

where for $B \in \{X, Y, M\}$ and $\ell \in \eta_B \equiv \{1, \ldots, n_B\}$, we define $\Psi_{B,\ell}$ as the tent function

$$\Psi_{B,\ell}(B) = \begin{cases} \frac{B-B_{\ell-1}}{B_\ell - B_{\ell-1}} & \text{, if } B \in [B_{\ell-1}, B_\ell] \\ \frac{B_{\ell+1}-B}{B_{\ell+1}-B_\ell} & \text{, if } B \in [B_\ell, B_{\ell+1}] \\ 0 & \text{, otherwise} \end{cases}.$$

Then, given a set of coefficients $\kappa \equiv \{\kappa_{ijk}\}_{i\in\eta_X,\, j\in\eta_Y,\, k\in\eta_M}$, we approximate the policy function for e by

$$e(s;\kappa) = \sum_{i\in\eta_X}\sum_{j\in\eta_Y}\sum_{k\in\eta_M} \kappa_{ijk}\Psi_{ijk}(s)\,. \tag{A.11}$$

From this policy function, we may then obtain the other implied policy functions as

$$Y'(s;\kappa) = F(e(s;\kappa)) ,$$

$$X'(s;\kappa) = (1-\delta)X + \psi Y'(s;\kappa) ,$$

$$\lambda(s;\kappa) = U'\left(Y'(s;\kappa) + \left(1 - \frac{\gamma}{1-\delta}\right)X - \gamma\left(1 - \frac{\psi}{1-\delta}\right)Y\right) ,$$

$$\Omega(s;\kappa) = [e(s;\kappa)]^{\varphi_e}\lambda(s;\kappa) .$$

Next, define

$$\Omega'(s,\nu_\mu;\kappa) = \Omega\left(X'(s;\kappa), Y'(s;\kappa), \tanh\left(\rho_\mu \tanh^{-1}(M) + \sqrt{2}\sigma_\mu\nu_\mu\right);\kappa\right)$$

as the value of Ω in the subsequent period given the current state s and the next-period shock innovation realization ν_μ. Then we may write the Euler equation as

$$R(s;\kappa) = 0 , \tag{A.12}$$

where

$$R(s;\kappa) \equiv \frac{1}{\sqrt{\pi}}\sqrt{\frac{1-M}{1+M}}\frac{Q(e(s;\kappa))}{\lambda(s;\kappa)}\Omega^e(s;\kappa) - 1 ,$$

$$\Omega^e(s;\kappa) \equiv \sqrt{\pi}\mathbb{E}[\Omega'(s,\nu_\mu;\kappa)] = \int_{-\infty}^{\infty}\exp\left\{-\nu_\mu^2\right\}\Omega'(s,\nu_\mu;\kappa)\,d\nu_\mu .$$

In practice, the integrals in Ω^e are computed using a Gauss-Hermite quadrature with 10 nodes.

Given κ, (A.12) will only hold at all points in the state space if (A.11) is indeed the true solution. We therefore choose κ to make $R(s;\kappa)$ as close to zero as possible throughout the state space. In practice, letting $S_{ijk} \equiv [X_{i-1}, X_{i+1}] \times [Y_{j-1}, Y_{j+1}] \times [M_{k-1}, M_{k+1}]$, we choose the $n_X n_Y n_M$ elements of κ to solve the $n_X n_Y n_M$ equations

$$\int_{S_{ijk}} \Psi_{ijk}(s)\,R(s;\kappa)\,ds = 0 , \quad i \in \eta_X,\ j \in \eta_Y,\ k \in \eta_M . \tag{A.13}$$

In practice, the integrals are approximated using a Gauss-Legendre quadrature with 3 nodes. For $\ell = 1,2,3$, let p_ℓ denote the ℓ-th Gauss-Legendre node and w_ℓ the associated weight, and for $B \in \{X, Y, M\}$ define $\Psi_{B,i,\ell} \equiv \Psi_{B,i}(\tilde{B}_{i,\ell})$, where

$$\tilde{B}_{i,\ell} \equiv \frac{B_{i+1} - B_{i-1}}{2}p_\ell + \frac{B_{i+1} + B_{i-1}}{2} .$$

For $\ell, m, n \in \{1, 2, 3\}$, define also $R_{ijk}^{\ell mn}(\kappa) \equiv R((\tilde{X}_{i,\ell}, \tilde{Y}_{j,m}, \tilde{M}_{k,n}); \kappa)$. Then given the quadrature approximation to the integral in (A.13), one can show that we may write (A.13) equivalently as the requirement that

$$\operatorname{vec}(\boldsymbol{W}_{ijk})' \operatorname{vec}(\boldsymbol{R}_{ijk}(\kappa)) = 0 , \quad i \in \eta_X,\, j \in \eta_Y,\, k \in \eta_M . \tag{A.14}$$

where $\boldsymbol{W}_{ijk}$ is the $3 \times 3 \times 3$ array whose (ℓ, m, n)-th entry is given by

$$w_\ell w_m w_n \Psi_{X,i,\ell} \Psi_{Y,j,m} \Psi_{M,k,n} ,$$

and $\boldsymbol{R}_{ijk}(\kappa)$ is the $3 \times 3 \times 3$ array whose (ℓ, m, n)-th entry is given by $R_{ijk}^{\ell mn}(\kappa)$. A non-linear solver is used to find the κ that makes equations (A.14) hold up to a desired tolerance.

For the results reported in the paper, we use $n_X = n_Y = 20$ and $n_M = 15$, for a total of 6,000 elements. Gridpoint bounds were chosen so as to make the system unlikely to hit them in practice. Gridpoints for X and Y were evenly spaced. Gridpoints for M were chosen as evenly spaced percentiles (ranging from 0.001 to 0.999) of the unconditional distribution for M, which puts a higher concentration of gridpoints in regions where M is more likely to be.

F Euler Errors

For a given value of the current state vector (X_t, Y_t, μ_t, z_t), the components of equation (18) are computed as follows. First, $C_{t-1} = X_t/(1-\delta) + [1 - \psi/(1-\delta)]Y_t$. Next, e_t is computed using the approximate policy function, which in turn is used to yield $Y_{t+1} = z_t e_t^\alpha$, $C_t = X_t + Y_{t+1}$, and $X_{t+1} = (1-\delta)X_t + \psi Y_{t+1}$.

From there, the expectation $\mathbb{E}_t[e_{t+1}^{\varphi_e} \lambda_{t+1}]$ is computed using a Monte Carlo approach as follows. First we draw N random values of the date-$(t+1)$ shock innovations, and use them to compute associated draws for (μ_{t+1}, z_{t+1}). Together with the values of X_{t+1} and Y_{t+1} obtained above, each of these draws gives a draw for the full state $(X_{t+1}, Y_{t+1}, \mu_{t+1}, z_{t+1})$ at $t+1$. From each of these draws, we compute the associated e_{t+1} using the approximate policy function, and then in turn obtain $Y_{t+2} = z_{t+1} e_{t+1}^\alpha$, $C_{t+1} = X_{t+1} + Y_{t+2}$ and $\lambda_{t+1} = (C_{t+1} - \gamma C_t)^{-\omega}$. Finally, we set $\mathbb{E}_t[e_{t+1}^{\varphi_e} \lambda_{t+1}]$ equal to the mean of the resulting value of $e_{t+1}^{\varphi_e} \lambda_{t+1}$ across the N draws. In practice, we use $N = 15{,}000$.

References

Aruoba, S. B., Fernández-Villaverde, J. & Rubio-Ramírez, J. F. (2006), 'Comparing solution methods for dynamic equilibrium economies', *Journal of Economic Dynamics and Control* **30**(12), 2477–2508.