Second Supplement to "Quantile Treatment Effects and Bootstrap Inference under Covariate-Adaptive Randomization": Strata Fixed Effects Quantile Regression Estimation and Additional Simulation Results

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Abstract

This paper gathers the theories for the strata fixed effects quantile regression estimator and additional simulation results. Section S.A describes the estimation, weighted bootstrap, and covariate-adaptive bootstrap inference procedures for the strata fixed effects quantile regression estimator. Sections S.B–S.D prove Theorems S.A.1–S.A.3, respectively. Section S.E contains the proofs of the technical lemmas. Section S.F contains additional simulation results.

Keywords: Bootstrap inference, quantile treatment effect

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S.A Quantile Regression with Strata Fixed Effects

The strata fixed effects estimator for the ATE is obtained by a linear regression of outcome Y_i on the treatment status A_i, controlling for strata dummies $\{1\{S_i = s\}_{s\in\mathcal{S}}\}$. Bugni, Canay, and Shaikh (2018) point out that, due to the Frisch-Waugh-Lovell theorem, this estimator is equal to the linear coefficient in the regression of Y_i on $\tilde{A}_i$, in which $\tilde{A}_i$ is the residual of the projection of A_i on the strata dummies. Unlike the expectation, the quantile operator is nonlinear. Therefore, we cannot consistently estimate QTEs by a linear QR of Y_i on A_i and strata dummies. Instead, based on the equivalence relationship, we propose to run the QR of Y_i on $\tilde{A}_i$. Formally, let $\tilde{A}_i = A_i - \hat{\pi}(S_i)$ and $\dot{\tilde{A}}_i = (1, \tilde{A}_i)'$, where $\hat{\pi}(s) = n_1(s)/n(s)$, $n_1(s) = \sum_{i=1}^n A_i 1\{S_i = s\}$, and $n(s) = \sum_{i=1}^n 1\{S_i = s\}$. Then, the strata fixed effects (SFE) estimator for the QTE is $\hat{\beta}_{sfe,1}(\tau)$, where

$$\hat{\beta}_{sfe}(\tau) \equiv \left(\hat{\beta}_{sfe,0}(\tau), \hat{\beta}_{sfe,1}(\tau)\right)' = \operatorname*{arg\,min}_{b=(b_0,b_1)'\in\Re^2} \sum_{i=1}^n \rho_\tau\left(Y_i - \dot{\tilde{A}}_i' b\right).$$

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Theorem S.A.1. *If Assumptions 1.1–1.3 and 2 hold and $p(s) > 0$ for $s \in \mathcal{S}$, then, uniformly over $\tau \in \Upsilon$,*

$$\sqrt{n}\left(\hat{\beta}_{sfe,1}(\tau) - q(\tau)\right) \rightsquigarrow \mathcal{B}_{sfe}(\tau), \text{ as } n \to \infty,$$

where $\mathcal{B}_{sfe}(\cdot)$ is a Gaussian process with covariance kernel $\Sigma_{sfe}(\cdot,\cdot)$. The expression for $\Sigma_{sfe}(\cdot,\cdot)$ can be found in the proof of this theorem.

In particular, the asymptotic variance for $\hat{\beta}_{sfe,1}(\tau)$ is

$$\zeta_Y^2(\pi,\tau) + \zeta_A'^2(\pi,\tau) + \zeta_S^2(\tau),$$

where $\zeta_Y^2(\pi,\tau)$ and $\zeta_S^2(\tau)$ are the same as those defined below Theorem 3.1,

$$\begin{aligned}\zeta_A'^2(\pi,\tau) =& \mathbb{E}\gamma(S)\Bigg[(m_1(S,\tau) - m_0(S,\tau))\left(\frac{1-\pi}{\pi f_1(q_1(\tau))} - \frac{\pi}{(1-\pi)f_0(q_0(\tau))}\right) \\ &+ q(\tau)\left(\frac{f_1(q_1(\tau)|S)}{f_1(q_1(\tau))} - \frac{f_0(q_0(\tau)|S)}{f_0(q_0(\tau))}\right)\Bigg]^2.\end{aligned}$$

Three remarks are in order. First, if the treatment assignment rule achieves strong balance, then $\zeta_A'^2(\pi,\tau) = 0$ and the asymptotic variances for $\hat{\beta}_1(\tau)$ and $\hat{\beta}_{sfe,1}(\tau)$ are the same. Second, if the treatment assignment rule does not achieve strong balance, then it is difficult to compare the asymptotic variances of $\hat{\beta}_1(\tau)$ and $\hat{\beta}_{sfe,1}(\tau)$. Based on our simulation results in Section S.F, the SFE estimator usually has a smaller standard error. Third, in order to analytically compute the asymptotic variance $\hat{\beta}_{sfe,1}(\tau)$, one needs to nonparametrically estimate not only the unconditional densities $f_j(\cdot)$ but also the conditional densities $f_j(\cdot|s)$ for $j = 0, 1$ and $s \in \mathcal{S}$. However, such difficulty can be avoided by the covariate-adaptive bootstrap inference considered in Section 5.

We can compute the weighted bootstrap counterpart of strata fixed effects estimator:

$$\hat{\beta}_{sfe}^w(\tau) = \arg\min_b \sum_{i=1}^n \xi_i \rho_\tau\left(Y_i - \dot{\tilde{A}}_i^{w\prime} b\right),$$

where $\dot{\tilde{A}}_i^w = (1, \tilde{A}_i^w)'$, $\tilde{A}_i^w = A_i - \hat{\pi}^w(S_i)$, and $\hat{\pi}^w(\cdot)$ is defined in Section 4. The second element of $\hat{\beta}_{sfe}^w(\tau)$ is our bootstrap estimator of the QTE.

Theorem S.A.2. *If Assumptions 1–3 hold and $p(s) > 0$ for all $s \in \mathcal{S}$, then uniformly over $\tau \in \Upsilon$ and conditionally on data,*

$$\sqrt{n}\left(\hat{\beta}_{sfe,1}^w(\tau) - \hat{\beta}_{sfe,1}(\tau)\right) \rightsquigarrow \tilde{\mathcal{B}}_{sfe}(\tau), \text{ as } n \to \infty,$$

where $\tilde{\mathcal{B}}_{sfe}(\tau)$ is a Gaussian process with covariance kernel being equal to that of $\mathcal{B}_{sfe}(\tau)$ defined in Theorem S.A.1 with $\gamma(s)$ being replaced by $\pi(1-\pi)$.

Similar to the SQR estimator, the weighted bootstrap fails to capture the cross-sectional dependence due to the covariate-adaptive randomization, and thus, overestimates the asymptotic variance of the SFE estimator.

We can also implement the covariate-adaptive bootstrap. Let

$$\hat{\beta}^*_{sfe}(\tau) = \arg\min_b \sum_{i=1}^n \rho_\tau \left(Y_i^* - \dot{\tilde{A}}_i^{*\prime} b \right),$$

where $\dot{\tilde{A}}_i^* = (1, \tilde{A}_i^*)'$, $\tilde{A}_i^* = A_i^* - \hat{\pi}^*(S_i^*)$, $\hat{\pi}^*(s) = \frac{n_1^*(s)}{n^*(s)}$, and $(Y_i^*, A_i^*, S_i^*)_{i=1}^n$ is the covariate-adaptive bootstrap sample generated via the procedure mentioned in Section 5. The the second element $\hat{\beta}^*_{sfe,1}(\tau)$ of $\hat{\beta}^*_{sfe}(\tau)$ is the covariate-adaptive SFE estimator.

Theorem S.A.3. *If Assumptions 1, 2, and 4 hold and $p(s) > 0$ for all $s \in \mathcal{S}$, then, uniformly over $\tau \in \Upsilon$ and conditionally on data,*

$$\sqrt{n}\left(\hat{\beta}^*_{sfe,1}(\tau) - \hat{q}(\tau) \right) \rightsquigarrow \mathcal{B}_{sfe}(\tau), \text{ as } n \to \infty.$$

Unlike the weighted bootstrap, the covariate-adaptive bootstrap can mimic the cross-sectional dependence, and thus, produces an asymptotically valid standard error for the SFE estimator.

S.B Proof of Theorem S.A.1

Define $\tilde{\beta}_1(\tau) = q(\tau)$, $\tilde{\beta}_0(\tau) = \pi q_1(\tau) + (1-\pi) q_0(\tau)$, $\tilde{\beta}(\tau) = (\tilde{\beta}_0(\tau), \tilde{\beta}_1(\tau))'$, and $\breve{A}_i = (1, A_i - \pi)'$. For arbitrary b_0 and b_1, let $u_0 = \sqrt{n}(b_0 - \tilde{\beta}_0(\tau))$, $u_1 = \sqrt{n}(b_1 - \tilde{\beta}_1(\tau))$, $u = (u_0, u_1)' \in \Re^2$, and

$$L_{sfe,n}(u,\tau) = \sum_{i=1}^n \left[\rho_\tau(Y_i - \breve{A}_i'\tilde{\beta}(\tau) - (\dot{\tilde{A}}_i' b - \breve{A}_i'\tilde{\beta}(\tau))) - \rho_\tau(Y_i - \breve{A}_i'\tilde{\beta}(\tau)) \right].$$

Then, by the change of variable, we have that

$$\sqrt{n}(\hat{\beta}_{sfe}(\tau) - \tilde{\beta}(\tau)) = \arg\min_u L_{sfe,n}(u,\tau).$$

Notice that $L_{sfe,n}(u,\tau)$ is convex in u for each τ and bounded in τ for each u. In the following, we aim to show that there exists

$$g_{sfe,n}(u,\tau) = -u' W_{sfe,n}(\tau) + \frac{1}{2} u' Q_{sfe}(\tau) u$$

such that (1) for each u,

$$\sup_{\tau \in \Upsilon} |L_{sfe,n}(u,\tau) - g_{sfe,n}(u,\tau) - h_{sfe,n}(\tau)| \xrightarrow{p} 0,$$

where $h_{sfe,n}(\tau)$ does not depend on u; (2) the maximum eigenvalue of $Q_{sfe}(\tau)$ is bounded from above and the minimum eigenvalue of $Q_{sfe}(\tau)$ is bounded away from 0 uniformly over $\tau \in \Upsilon$; (3) $W_{sfe,n}(\tau) \rightsquigarrow \tilde{\mathcal{B}}(\tau)$ uniformly over $\tau \in \Upsilon$ for some $\tilde{\mathcal{B}}(\tau)$.[1] Then by Kato (2009, Theorem 2), we have

$$\sqrt{n}(\hat{\beta}_{sfe}(\tau) - \tilde{\beta}(\tau)) = [Q_{sfe}(\tau)]^{-1} W_{sfe,n}(\tau) + r_{sfe,n}(\tau),$$

where $\sup_{\tau \in \Upsilon} ||r_{sfe,n}(\tau)|| = o_p(1)$. In addition, by (3), we have, uniformly over $\tau \in \Upsilon$,

$$\sqrt{n}(\hat{\beta}_{sfe}(\tau) - \tilde{\beta}(\tau)) \rightsquigarrow [Q_{sfe}(\tau)]^{-1} \tilde{\mathcal{B}}(\tau) \equiv \mathcal{B}(\tau).$$

The second element of $\mathcal{B}(\tau)$ is $\mathcal{B}_{sfe}(\tau)$ stated in Theorem S.A.1. Next, we prove requirements (1)–(3) in three steps.

Step 1. By Knight's identity (Knight, 1998), we have

$$\begin{aligned}
& L_{sfe,n}(u,\tau) \\
= & -\sum_{i=1}^{n} (\dot{\breve{A}}_i'(\tilde{\beta}(\tau) + \frac{u}{\sqrt{n}}) - \breve{A}_i'\tilde{\beta}(\tau)) \left(\tau - 1\{Y_i \leq \dot{\breve{A}}_i'\tilde{\beta}(\tau)\}\right) \\
& + \sum_{i=1}^{n} \int_0^{\dot{\breve{A}}_i'(\tilde{\beta}(\tau)+\frac{u}{\sqrt{n}}) - \breve{A}_i'\tilde{\beta}(\tau)} \left(1\{Y_i - \dot{\breve{A}}_i'\tilde{\beta}(\tau) \leq v\} - 1\{Y_i - \dot{\breve{A}}_i'\tilde{\beta}(\tau) \leq 0\}\right) dv \\
\equiv & - L_{1,n}(u,\tau) + L_{2,n}(u,\tau).
\end{aligned}$$

Step 1.1. We first consider $L_{1,n}(u,\tau)$. Note that $\tilde{\beta}_1(\tau) = q(\tau)$ and

$$\begin{aligned}
& L_{1,n}(u,\tau) \\
= & \sum_{i=1}^{n} \sum_{s \in \mathcal{S}} A_i 1\{S_i = s\} \left(\frac{u_0}{\sqrt{n}} + (1 - \hat{\pi}(s)) \frac{u_1}{\sqrt{n}} + (\pi - \hat{\pi}(s)) q(\tau) \right) (\tau - 1\{Y_i(1) \leq q_1(\tau)\}) \\
& + \sum_{i=1}^{n} \sum_{s \in \mathcal{S}} (1 - A_i) 1\{S_i = s\} \left(\frac{u_0}{\sqrt{n}} - \hat{\pi}(s) \frac{u_1}{\sqrt{n}} + (\pi - \hat{\pi}(s)) q(\tau) \right) (\tau - 1\{Y_i(0) \leq q_0(\tau)\}) \\
\equiv & L_{1,1,n}(u,\tau) + L_{1,0,n}(u,\tau). \qquad \text{(S.B.1)}
\end{aligned}$$

Let $\iota_1 = (1, 1 - \pi)'$ and $\iota_0 = (1, -\pi)'$. Note that $\hat{\pi}(s) - \pi = \frac{D_n(s)}{n(s)}$. Then, for $L_{1,1,n}(u,\tau)$, we have

$$\begin{aligned}
& L_{1,1,n}(u,\tau) \\
= & \sum_{i=1}^{n} \sum_{s \in \mathcal{S}} A_i 1\{S_i = s\} \left[\frac{u'\iota_1}{\sqrt{n}} + (\pi - \hat{\pi}(s)) \left(q(\tau) + \frac{u_1}{\sqrt{n}} \right) \right] (\tau - 1\{Y_i(1) \leq q_1(\tau)\})
\end{aligned}$$

[1] We abuse the notation and denote the weak limit of $W_{sfe,n}(\tau)$ as $\tilde{\mathcal{B}}(\tau)$. This limit is different from the weak limit of $W_n(\tau)$ in the proof of Theorem 3.1.

$$
\begin{aligned}
=&\frac{u'\iota_1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{s\in\mathcal{S}}A_i1\{S_i=s\}\left(\tau-1\{Y_i(1)\le q_1(\tau)\}\right)\\
&-\sum_{s\in\mathcal{S}}\frac{D_n(s)}{\sqrt{n}}\frac{u_1}{n(s)}\sum_{i=1}^{n}A_i1\{S_i=s\}\left(\tau-1\{Y_i(1)\le q_1(\tau)\}\right)\\
&+\sum_{s\in\mathcal{S}}(\pi-\hat{\pi}(s))q(\tau)\sum_{i=1}^{n}A_i1\{S_i=s\}\left(\tau-1\{Y_i(1)\le q_1(\tau)\}\right)\\
=&\sum_{s\in\mathcal{S}}\frac{u'\iota_1}{\sqrt{n}}\sum_{i=1}^{n}\Big[A_i1\{S_i=s\}\eta_{i,1}(s,\tau)+(A_i-\pi)1\{S_i=s\}m_1(s,\tau)+\pi1\{S_i=s\}m_1(s,\tau)\Big]\\
&-\sum_{s\in\mathcal{S}}\frac{D_n(s)}{\sqrt{n}}\frac{u_1}{n(s)}\sum_{i=1}^{n}\Big[A_i1\{S_i=s\}\eta_{i,1}(s,\tau)+(A_i-\pi)1\{S_i=s\}m_1(s,\tau)+\pi1\{S_i=s\}m_1(s,\tau)\Big]+h_{1,1}(\tau)\\
=&\sum_{s\in\mathcal{S}}\frac{u'\iota_1}{\sqrt{n}}\sum_{i=1}^{n}\Big[A_i1\{S_i=s\}\eta_{i,1}(s,\tau)+(A_i-\pi)1\{S_i=s\}m_1(s,\tau)+\pi1\{S_i=s\}m_1(s,\tau)\Big]\\
&-\sum_{s\in\mathcal{S}}\frac{u_1D_n(s)\pi m_1(s,\tau)}{\sqrt{n}}+h_{1,1}(\tau)+R_{sfe,1,1}(u,\tau),
\end{aligned}
\tag{S.B.2}
$$

where

$$
h_{1,1}(\tau)=\sum_{s\in\mathcal{S}}(\pi-\hat{\pi}(s))q(\tau)\sum_{i=1}^{n}A_i1\{S_i=s\}\left(\tau-1\{Y_i(1)\le q_1(\tau)\}\right)
$$

and

$$
R_{sfe,1,1}(u,\tau)=-\sum_{s\in\mathcal{S}}\frac{u_1D_n(s)}{\sqrt{n}n(s)}\sum_{i=1}^{n}\Big[A_i1\{S_i=s\}\eta_{i,1}(s,\tau)+(A_i-\pi)1\{S_i=s\}m_1(s,\tau)\Big].
$$

By the same argument in Lemma E.2 and Assumption 1.3, we have for every $s\in\mathcal{S}$,

$$
\sup_{\tau\in\Upsilon}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}A_i1\{S_i=s\}\eta_{i,1}(s,\tau)\right|=O_p(1)
\tag{S.B.3}
$$

and

$$
\sup_{\tau\in\Upsilon}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big[(A_i-\pi)1\{S_i=s\}m_1(s,\tau)\Big]\right|=\sup_{\tau\in\Upsilon}\left|\frac{D_n(s)m_1(s,\tau)}{\sqrt{n}}\right|=O_p(1).
$$

In addition, note that $n(s)/n\xrightarrow{p}p(s)$. Therefore,

$$
\sup_{\tau\in\Upsilon}|R_{sfe,1,1}(u,\tau)|=O_p(\frac{1}{\sqrt{n}})=o_p(1).
$$

Similarly, we have

$$
\begin{aligned}
& L_{1,0,n}(u,\tau) \\
=& \sum_{s\in\mathcal{S}} \frac{u'\iota_0}{\sqrt{n}} \sum_{i=1}^{n} \Big[(1-A_i)1\{S_i=s\}\eta_{i,0}(s,\tau) - (A_i-\pi)1\{S_i=s\}m_0(s,\tau) + (1-\pi)1\{S_i=s\}m_0(s,\tau)\Big] \\
& - \sum_{s\in\mathcal{S}} \frac{u_1 D_n(s)(1-\pi)m_0(s,\tau)}{\sqrt{n}} + h_{1,0}(\tau) + R_{sfe,1,0}(u,\tau), \qquad \text{(S.B.4)}
\end{aligned}
$$

where

$$
h_{1,0}(\tau) = \sum_{s\in\mathcal{S}} (\pi - \hat{\pi}(s)) q(\tau) \sum_{i=1}^{n} (1-A_i)1\{S_i=s\}\left(\tau - 1\{Y_i(0)\le q_0(\tau)\}\right),
$$

$$
R_{sfe,1,0}(u,\tau) = -\sum_{s\in\mathcal{S}} \frac{u_1 D_n(s)}{\sqrt{n}n(s)} \sum_{i=1}^{n} \Big[(1-A_i)1\{S_i=s\}\eta_{i,0}(\tau) - (A_i-\pi)1\{S_i=s\}m_0(s,\tau)\Big],
$$

and

$$
\sup_{\tau\in\Upsilon} |R_{sfe,1,0}(\tau)| = O_p\Big(\frac{1}{\sqrt{n}}\Big) = o_p(1).
$$

Combining (S.B.1), (S.B.2), (S.B.4) and letting $\iota_2 = (1, 1-2\pi)'$, we have

$$
\begin{aligned}
L_{1,n}(u,\tau) =& \frac{1}{\sqrt{n}} \sum_{s\in\mathcal{S}} \sum_{i=1}^{n} \Big[u'\iota_1 A_i 1\{S_i=s\}\eta_{i,1}(s,\tau) + u'\iota_0(1-A_i)1\{S_i=s\}\eta_{i,0}(s,\tau)\Big] \\
& + \sum_{s\in\mathcal{S}} u'\iota_2 \frac{D_n(s)}{\sqrt{n}} \left(m_1(s,\tau) - m_0(s,\tau)\right) \\
& + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(u'\iota_1 \pi m_1(S_i,\tau) + u'\iota_0(1-\pi)m_0(S_i,\tau)\right) \\
& + R_{sfe,1,1}(u,\tau) + R_{sfe,1,0}(u,\tau) + h_{1,1}(\tau) + h_{1,0}(\tau). \qquad \text{(S.B.5)}
\end{aligned}
$$

Step 1.2. Next, we consider $L_{2,n}(u,\tau)$. Denote $E_n(s) = \sqrt{n}(\hat{\pi}(s) - \pi)$. Then,

$$
\{E_n(s)\}_{s\in\mathcal{S}} = \left\{\frac{D_n(s)}{\sqrt{n}} \frac{n}{n(s)}\right\}_{s\in\mathcal{S}} \rightsquigarrow \mathcal{N}(0, \Sigma_D') = O_p(1),
$$

where $\Sigma_D' = \text{diag}(\gamma(s)/p(s) : s\in\mathcal{S})$. In addition,

$L_{2,n}(u,\tau)$

$$
\begin{aligned}
=&\sum_{s\in\mathcal{S}}\sum_{i=1}^{n}A_i1\{S_i=s\}\int_0^{\frac{u'\iota_1}{\sqrt{n}}-\frac{E_n(s)}{\sqrt{n}}\left(q(\tau)+\frac{u_1}{\sqrt{n}}\right)}\left(1\{Y_i(1)\leq q_1(\tau)+v\}-1\{Y_i(1)\leq q_1(\tau)\}\right)dv\\
&+\sum_{s\in\mathcal{S}}\sum_{i=1}^{n}(1-A_i)1\{S_i=s\}\int_0^{\frac{u'\iota_0}{\sqrt{n}}-\frac{E_n(s)}{\sqrt{n}}\left(q(\tau)+\frac{u_1}{\sqrt{n}}\right)}\left(1\{Y_i(0)\leq q_0(\tau)+v\}-1\{Y_i(0)\leq q_0(\tau)\}\right)dv\\
\equiv&L_{2,1,n}(u,\tau)+L_{2,0,n}(u,\tau). \qquad \text{(S.B.6)}
\end{aligned}
$$

By the same argument in (A.1), we have

$$
\begin{aligned}
L_{2,1,n}(u,\tau)\overset{d}{=}&\sum_{s\in\mathcal{S}}\sum_{i=N(s)+1}^{N(s)+n_1(s)}\int_0^{\frac{u'\iota_1}{\sqrt{n}}-\frac{E_n(s)}{\sqrt{n}}\left(q(\tau)+\frac{u_1}{\sqrt{n}}\right)}\left(1\{Y_i^s(1)\leq q_1(\tau)+v\}-1\{Y_i^s(1)\leq q_1(\tau)\}\right)dv\\
\equiv&\sum_{s\in\mathcal{S}}\left[\Gamma_n^s(N(s)+n_1(s),\tau,E_n(s))-\Gamma_n^s(N(s),\tau,E_n(s))\right], \qquad \text{(S.B.7)}
\end{aligned}
$$

where

$$
\Gamma_n^s(k,\tau,e)=\sum_{i=1}^{k}\int_0^{\frac{u'\iota_1-e(q(\tau)+\frac{u_1}{\sqrt{n}})}{\sqrt{n}}}\left(1\{Y_i^s(1)\leq q_1(\tau)+v\}-1\{Y_i^s(1)\leq q_1(\tau)\}\right)dv.
$$

We want to show, for some any sufficiently large constant M,

$$
\sup_{0<t\leq 1,\tau\in\Upsilon,|e|\leq M}|\Gamma_n^s(\lfloor nt\rfloor,\tau,e)-\mathbb{E}\Gamma_n^s(\lfloor nt\rfloor,\tau,e)|=o_p(1). \qquad \text{(S.B.8)}
$$

By the same argument in (A.2), it suffices to show that

$$
\sup_{\tau\in\Upsilon,|e|\leq M}n||\mathbb{P}_n-\mathbb{P}||_{\mathcal{F}}=o_p(1),
$$

where

$$
\mathcal{F}=\left\{\int_0^{\frac{u'\iota_1-e(q(\tau)+\frac{u_1}{\sqrt{n}})}{\sqrt{n}}}\left(1\{Y_i^s(1)\leq q_1(\tau)+v\}-1\{Y_i^s(1)\leq q_1(\tau)\}\right)dv:\tau\in\Upsilon,|e|\leq M\right\}
$$

with an envelope $F=\frac{|u_0|+|u_1|+M\sup_{\tau\in\Upsilon}|q(\tau)|+\frac{|u_1|}{\sqrt{n}}}{\sqrt{n}}$. Note that

$$
\begin{aligned}
\sup_{f\in\mathcal{F}}\mathbb{E}f^2\leq&\sup_{\tau\in\Upsilon}\mathbb{E}\left[\frac{|u_0|+|u_1|+M|q(\tau)|+\frac{|u_1|}{\sqrt{n}}}{\sqrt{n}}1\left\{|Y_i^s(1)-q_1(\tau)|\leq\frac{|u_0|+|u_1|+M|q(\tau)|+\frac{|u_1|}{\sqrt{n}}}{\sqrt{n}}\right\}\right]^2\\
\lesssim&n^{-3/2},
\end{aligned}
$$

and $\mathcal{F}$ is a VC-class with a fixed VC index. Then, by Chernozhukov, Chetverikov, and Kato (2014, Corollary 5.1),

$$\mathbb{E}\sup_{\tau\in\Upsilon,|e|\leq M}|\Gamma_n^s(n,\tau,e)-\mathbb{E}\Gamma_n^s(n,\tau,e)|=n||\mathbb{P}_n-\mathbb{P}||_{\mathcal{F}}\lesssim n\left[\sqrt{\frac{\log(n)}{n^{5/2}}}+\frac{\log(n)}{n^{3/2}}\right]=o(1). \quad \text{(S.B.9)}$$

In addition, we have

$$\begin{aligned}\mathbb{E}\Gamma_n^s(\lfloor nt\rfloor,\tau,e)=&\lfloor nt\rfloor\int_0^{\frac{u'\iota_1-e(q(\tau)+\frac{u_1}{\sqrt{n}})}{\sqrt{n}}}[F_1(q_1(\tau)+v|s)-F_1(q_1(\tau)|s)]dv\\=&t\frac{f_1(q_1(\tau)|s)}{2}(u'\iota_1-eq(\tau))^2+o(1),\end{aligned} \quad \text{(S.B.10)}$$

where $F_j(\cdot|s)$ and $f_j(\cdot|s)$, $j=0,1$ are the conditional CDF and PDF for $Y(j)$ given $S=s$, respectively, and the $o(1)$ term holds uniformly over $\{\tau\in\Upsilon,|e|\leq M\}$. Combining (S.B.8) and (S.B.10) with the fact that $\frac{n_1(s)}{n}\xrightarrow{p}\pi p(s)$, we have

$$\begin{aligned}L_{2,1,n}(u,\tau)=&\sum_{s\in\mathcal{S}}\pi p(s)\frac{f_1(q_1(\tau)|s)}{2}(u'\iota_1-E_n(s)q(\tau))^2+R'_{sfe,2,1}(u,\tau)\\=&\frac{\pi f_1(q_1(\tau))}{2}(u'\iota_1)^2-\sum_{s\in\mathcal{S}}f_1(q_1(\tau)|s)\frac{\pi D_n(s)u'\iota_1}{\sqrt{n}}q(\tau)+h_{2,1}(\tau)+R_{sfe,2,1}(u,\tau),\end{aligned} \quad \text{(S.B.11)}$$

where

$$\sup_{\tau\in\Upsilon}|R'_{sfe,2,1}(u,\tau)|=o_p(1),\quad \sup_{\tau\in\Upsilon}|R_{sfe,2,1}(u,\tau)|=o_p(1),$$

and

$$h_{2,1}(\tau)=\sum_{s\in\mathcal{S}}\frac{\pi f_1(q_1(\tau)|s)}{2}p(s)E_n^2(s)\tilde{\beta}_1^2(\tau).$$

Similarly, we have

$$\begin{aligned}L_{2,0,n}(u,\tau)=&\frac{(1-\pi)f_0(q_0(\tau))}{2}(u'\iota_0)^2-\sum_{s\in\mathcal{S}}(1-\pi)f_0(q_0(\tau)|s)\frac{D_n(s)u'\iota_0}{\sqrt{n}}q(\tau)\\&+h_{2,0}(\tau)+R_{sfe,2,0}(u,\tau),\end{aligned} \quad \text{(S.B.12)}$$

where

$$\sup_{\tau\in\Upsilon}|R_{sfe,2,0}(u,\tau)|=o_p(1)\quad\text{and}\quad h_{2,0}(\tau)=\sum_{s\in\mathcal{S}}\frac{(1-\pi)f_0(q_0(\tau)|s)}{2}p(s)E_n^2(s)\tilde{\beta}_1^2(\tau).$$

Combining (S.B.6), (S.B.11), and (S.B.12), we have

$$
\begin{aligned}
L_{2,n}(u,\tau) =& \frac{1}{2}u'Q_{sfe}(\tau)u - \sum_{s\in\mathcal{S}} q(\tau)\left[f_1(q_1(\tau)|s)\pi u'\iota_1 + f_0(q_0(\tau)|s)(1-\pi)u'\iota_0\right]\frac{D_n(s)}{\sqrt{n}} \\
&+ R_{sfe,2,1}(u,\tau) + R_{sfe,2,0}(u,\tau) + h_{2,1}(\tau) + h_{2,0}(\tau). \qquad \text{(S.B.13)}
\end{aligned}
$$

where

$$
\begin{aligned}
Q_{sfe} =& \pi f_1(q_1(\tau))\iota_1\iota_1' + (1-\pi)f_0(q_0(\tau))\iota_0\iota_0' \\
=& \begin{pmatrix} \pi f_1(q_1(\tau)) + (1-\pi)f_0(q_0(\tau)) & \pi(1-\pi)(f_1(q_1(\tau)) - f_0(q_0(\tau))) \\ \pi(1-\pi)(f_1(q_1(\tau)) - f_0(q_0(\tau))) & \pi(1-\pi)((1-\pi)f_1(q_1(\tau)) + \pi f_0(q_0(\tau))) \end{pmatrix}.
\end{aligned}
$$

Step 1.3. Last, by combining (S.B.5) and (S.B.13), we have

$$
L_{sfe,n}(u,\tau) = -u'W_{sfe,n}(\tau) + \frac{1}{2}u'Q_{sfe}(\tau)u + R_{sfe}(u,\tau) + h_{sfe,n}(\tau),
$$

where

$$
\begin{aligned}
& W_{sfe,n}(\tau) \\
=& \frac{1}{\sqrt{n}}\sum_{s\in\mathcal{S}}\sum_{i=1}^{n}\Big[\iota_1 A_i 1\{S_i = s\}\eta_{i,1}(s,\tau) + \iota_0(1-A_i)1\{S_i = s\}\eta_{i,0}(s,\tau)\Big] \\
&+ \sum_{s\in\mathcal{S}}\Big\{\iota_2\left(m_1(s,\tau) - m_0(s,\tau)\right) + q(\tau)\Big[f_1(q_1(\tau)|s)\pi\iota_1 + f_0(q_0(\tau)|s)(1-\pi)\iota_0\Big]\Big\}\frac{D_n(s)}{\sqrt{n}} \\
&+ \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\iota_1\pi m_1(S_i,\tau) + \iota_0(1-\pi)m_0(S_i,\tau)\right) \\
\equiv& W_{sfe,n,1}(\tau) + W_{sfe,n,2}(\tau) + W_{sfe,n,3}(\tau), \qquad \text{(S.B.14)}
\end{aligned}
$$

$$
R_{sfe}(u,\tau) = R_{sfe,1,1}(u,\tau) + R_{sfe,1,0}(u,\tau) + R_{sfe,2,1}(u,\tau) + R_{sfe,2,0}(u,\tau)
$$

such that $\sup_{\tau\in\Upsilon}|R_{sfe}(u,\tau)| = o_p(1)$, and

$$
h_{sfe,n}(\tau) = h_{1,1}(\tau) + h_{1,0}(\tau) + h_{2,1}(\tau) + h_{2,0}(\tau).
$$

This concludes the proof of Step 1.

Step 2. Note that $\det(Q_{sfe}(\tau)) = \pi(1-\pi)f_0(q_0(\tau))f_1(q_1(\tau))$, which is bounded and bounded away from zero. In addition, it can be shown that the two eigenvalues of $Q_{sfe}(\tau)$ are nonnegative. This leads to the desired result.

Step 3. Lemma S.E.1 establishes the weak convergence that

$$(W_{sfe,1,n}(\tau), W_{sfe,2,n}(\tau), W_{sfe,3,n}(\tau)) \rightsquigarrow (\mathcal{B}_{sfe,1}(\tau), \mathcal{B}_{sfe,2}(\tau), \mathcal{B}_{sfe,3}(\tau)),$$

where $(\mathcal{B}_{sfe,1}(\tau), \mathcal{B}_{sfe,2}(\tau), \mathcal{B}_{sfe,3}(\tau))$ are three independent two-dimensional Gaussian processes with covariance kernels $\Sigma_1(\tau_1,\tau_2)$, $\Sigma_2(\tau_1,\tau_2)$, and $\Sigma_3(\tau_1,\tau_2)$, respectively. Therefore, uniformly over $\tau \in \Upsilon$,

$$W_{sfe,n}(\tau) \rightsquigarrow \tilde{\mathcal{B}}(\tau),$$

where $\tilde{\mathcal{B}}(\tau)$ is a two-dimensional Gaussian process with covariance kernel

$$\tilde{\Sigma}(\tau_1,\tau_2) = \sum_{j=1}^{3} \Sigma_j(\tau_1,\tau_2).$$

Consequently,

$$\sqrt{n}(\hat{\beta}_{sfe}(\tau) - \tilde{\beta}(\tau)) \rightsquigarrow \mathcal{B}(\tau) \equiv Q_{sfe}^{-1}(\tau)\tilde{\mathcal{B}}(\tau),$$

where $\Sigma(\tau_1,\tau_2)$, the covariance kernel of $\mathcal{B}(\tau)$, has the expression that

$$\begin{aligned}
&\Sigma(\tau_1,\tau_2)\\
=&Q_{sfe}^{-1}(\tau_1)\tilde{\Sigma}(\tau_1,\tau_2)Q_{sfe}^{-1}(\tau_2)\\
=&\Bigg\{\frac{1}{\pi f_1(q_1(\tau_1))f_1(q_1(\tau_2))}\left[\min(\tau_1,\tau_2) - \tau_1\tau_2 - \mathbb{E}m_1(S,\tau_1)m_1(S,\tau_2)\right]\begin{pmatrix}\pi^2 & \pi\\ \pi & 1\end{pmatrix}\\
&+\frac{1}{(1-\pi)f_0(q_0(\tau_1))f_0(q_0(\tau_2))}\left[\min(\tau_1,\tau_2) - \tau_1\tau_2 - \mathbb{E}m_0(S,\tau_1)m_0(S,\tau_2)\right]\begin{pmatrix}(1-\pi)^2 & \pi-1\\ \pi-1 & 1\end{pmatrix}\Bigg\}\\
&+\Bigg\{\mathbb{E}\gamma(S)\Bigg[(m_1(S,\tau_1)-m_0(S,\tau_1))\begin{pmatrix}\frac{\pi}{f_0(q_0(\tau_1))}+\frac{1-\pi}{f_1(q_1(\tau_1))}\\ \frac{1-\pi}{\pi f_1(q_1(\tau_1))}-\frac{\pi}{(1-\pi)f_0(q_0(\tau_1))}\end{pmatrix}+q(\tau_1)\frac{f_1(q_1(\tau_1)|S)}{f_1(q_1(\tau_1))}\begin{pmatrix}\pi\\ 1\end{pmatrix}\\
&+q(\tau_1)\frac{f_0(q_0(\tau_1)|S)}{f_0(q_0(\tau_1))}\begin{pmatrix}1-\pi\\ -1\end{pmatrix}\Bigg]\times\Bigg[(m_1(S,\tau_2)-m_0(S,\tau_2))\begin{pmatrix}\frac{\pi}{f_0(q_0(\tau_2))}+\frac{1-\pi}{f_1(q_1(\tau_2))}\\ \frac{1-\pi}{\pi f_1(q_1(\tau_2))}-\frac{\pi}{(1-\pi)f_0(q_0(\tau_2))}\end{pmatrix}\\
&+q(\tau_2)\frac{f_1(q_1(\tau_2)|S)}{f_1(q_1(\tau_2))}\begin{pmatrix}\pi\\ 1\end{pmatrix}+q(\tau_2)\frac{f_0(q_0(\tau_2)|S)}{f_0(q_0(\tau_2))}\begin{pmatrix}1-\pi\\ -1\end{pmatrix}\Bigg]\Bigg\}\\
&+\Bigg\{\mathbb{E}\left[\frac{m_1(S,\tau_1)}{f_1(q_1(\tau_1))}\begin{pmatrix}\pi\\ 1\end{pmatrix}+\frac{m_0(S,\tau_1)}{f_0(q_0(\tau_1))}\begin{pmatrix}1-\pi\\ -1\end{pmatrix}\right]\left[\frac{m_1(S,\tau_2)}{f_1(q_1(\tau_2))}\begin{pmatrix}\pi\\ 1\end{pmatrix}+\frac{m_0(S,\tau_2)}{f_0(q_0(\tau_2))}\begin{pmatrix}1-\pi\\ -1\end{pmatrix}\right]'\Bigg\}.
\end{aligned}$$

By checking the $(2,2)$-element of $\Sigma(\tau_1,\tau_2)$, we have

$$\Sigma_{sfe}(\tau_1,\tau_2)$$

$$
\begin{aligned}
=&\frac{\min(\tau_1,\tau_2)-\tau_1\tau_2-\mathbb{E}m_1(S,\tau_1)m_1(S,\tau_2)}{\pi f_1(q_1(\tau_1))f_1(q_1(\tau_2))}+\frac{\min(\tau_1,\tau_2)-\tau_1\tau_2-\mathbb{E}m_0(S,\tau_1)m_0(S,\tau_2)}{(1-\pi)f_0(q_0(\tau_1))f_0(q_0(\tau_2))}\\
&+\mathbb{E}\gamma(S)\left[(m_1(S,\tau_1)-m_0(S,\tau_1))\left(\frac{1-\pi}{\pi f_1(q_1(\tau_1))}-\frac{\pi}{(1-\pi)f_0(q_0(\tau_1))}\right)+q(\tau_1)\left(\frac{f_1(q(\tau_1)|S)}{f_1(q_1(\tau_1))}-\frac{f_0(q(\tau_1)|S)}{f_0(q_0(\tau_1))}\right)\right]\\
&\times\left[(m_1(S,\tau_2)-m_0(S,\tau_2))\left(\frac{1-\pi}{\pi f_1(q_1(\tau_2))}-\frac{\pi}{(1-\pi)f_0(q_0(\tau_2))}\right)+q(\tau_2)\left(\frac{f_1(q(\tau_2)|S)}{f_1(q_2(\tau_2))}-\frac{f_0(q(\tau_2)|S)}{f_0(q_0(\tau_2))}\right)\right]\\
&+\mathbb{E}\left[\frac{m_1(S,\tau_1)}{f_1(q_1(\tau_1))}-\frac{m_0(S,\tau_1)}{f_0(q_0(\tau_1))}\right]\left[\frac{m_1(S,\tau_2)}{f_1(q_1(\tau_2))}-\frac{m_0(S,\tau_2)}{f_0(q_0(\tau_2))}\right].
\end{aligned}
$$

S.C Proof of Theorem S.A.2

Note that

$$\sqrt{n}(\hat{\beta}^w_{sfe}(\tau)-\tilde{\beta}(\tau))=\arg\min_u L^w_{sfe,n}(u,\tau),$$

where

$$L^w_{sfe,n}(u,\tau)=\sum_{i=1}^n \xi_i\left[\rho_\tau(Y_i-\dot{\tilde{A}}_i^{w\prime}(\tilde{\beta}(\tau)+\frac{u}{\sqrt{n}}))-\rho_\tau(Y_i-\breve{A}_i'\tilde{\beta}(\tau))\right],$$

$\dot{\tilde{A}}_i^w=(1,\tilde{A}_i^w)'$, $\tilde{A}_i^w=A_i-\hat{\pi}^w(S_i)$, and

$$\hat{\pi}^w(s)=\frac{\sum_{i=1}^n\xi_iA_i1\{S_i=s\}}{\sum_{i=1}^n\xi_i1\{S_i=s\}}.$$

Similar to the proof of Theorem S.A.1, we divide the proof into two steps. In the first step, we show that there exists

$$g^w_{sfe,n}(u,\tau)=-u'W^w_{sfe,n}(\tau)+\frac{1}{2}u'Q_{sfe}(\tau)u$$

and $h^w_{sfe,n}(\tau)$ independent of u such that for each u

$$\sup_{\tau\in\Upsilon}|L^w_{sfe,n}(u,\tau)-g^w_{sfe,n}(u,\tau)-h^w_{sfe,n}(\tau)|\xrightarrow{p}0.$$

In addition, we will show that $\sup_{\tau\in\Upsilon}||W^w_{sfe,n}(\tau)||=O_p(1)$. Then, by Kato (2009, Theorem 2), we have

$$\sqrt{n}(\hat{\beta}^w_{sfe}(\tau)-\tilde{\beta}(\tau))=[Q_{sfe}(\tau)]^{-1}W^w_{sfe,n}(\tau)+R^w_{sfe,n}(\tau),$$

where

$$\sup_{\tau\in\Upsilon}||R^w_{sfe,n}(\tau)||=o_p(1).$$

In the second step, we show that, conditionally on data,

$$\sqrt{n}(\hat{\beta}^w_{sfe,1}(\tau) - \hat{\beta}_{sfe,1}(\tau)) \rightsquigarrow \tilde{\mathcal{B}}_{sfe}(\tau).$$

Step 1. Following Step 1 in the proof of Theorem S.A.1, we have

$$L^w_{sfe,n}(u,\tau) \equiv -L^w_{1,n}(u,\tau) + L^w_{2,n}(u,\tau),$$

where

$$\begin{aligned}
&L^w_{1,n}(u,\tau)\\
=&\sum_{i=1}^n \sum_{s\in\mathcal{S}} \xi_i A_i 1\{S_i = s\}\left(\frac{u_0}{\sqrt{n}} + (1-\hat{\pi}^w(s))\frac{u_1}{\sqrt{n}} + (\pi - \hat{\pi}^w(s))q(\tau)\right)(\tau - 1\{Y_i \leq q_1(\tau)\})\\
&+\sum_{i=1}^n \sum_{s\in\mathcal{S}} \xi_i (1-A_i) 1\{S_i = s\}\left(\frac{u_0}{\sqrt{n}} - \hat{\pi}^w(s)\frac{u_1}{\sqrt{n}} + (\pi - \hat{\pi}^w(s))q(\tau)\right)(\tau - 1\{Y_i \leq q_0(\tau)\})\\
\equiv& L^w_{1,1,n}(u,\tau) + L^w_{1,0,n}(u,\tau),
\end{aligned}$$

$$\begin{aligned}
&L^w_{2,n}(u,\tau)\\
=&\sum_{s\in\mathcal{S}} \sum_{i=1}^n \xi_i A_i 1\{S_i = s\} \int_0^{\frac{u'\iota_1}{\sqrt{n}} - \frac{E^w_n(s)}{\sqrt{n}}\left(q(\tau) + \frac{u_1}{\sqrt{n}}\right)} (1\{Y_i \leq q_1(\tau) + v\} - 1\{Y_i \leq q_1(\tau)\})\,dv\\
&+\sum_{s\in\mathcal{S}} \sum_{i=1}^n \xi_i (1-A_i) 1\{S_i = s\} \int_0^{\frac{u'\iota_0}{\sqrt{n}} - \frac{E^w_n(s)}{\sqrt{n}}\left(q(\tau) + \frac{u_1}{\sqrt{n}}\right)} (1\{Y_i \leq q_0(\tau) + v\} - 1\{Y_i \leq q_0(\tau)\})\,dv\\
\equiv& L^w_{2,1,n}(u,\tau) + L^w_{2,0,n}(u,\tau),
\end{aligned}$$

and $E^w_n(s) = \sqrt{n}(\hat{\pi}^w(s) - \pi)$.

Step 1.1. Recall that $\iota_1 = (1, 1-\pi)'$ and $\iota_0 = (1, -\pi)'$. In addition, denote $\hat{\pi}^w(s) - \pi = \frac{D^w_n(s)}{n^w(s)}$, where

$$D^w_n(s) = \sum_{i=1}^n \xi_i (A_i - \pi) 1\{S_i = s\} \quad \text{and} \quad n^w(s) = \sum_{i=1}^n \xi_i 1\{S_i = s\}.$$

Then, we have

$$\begin{aligned}
&L^w_{1,1,n}(u,\tau)\\
=&\sum_{s\in\mathcal{S}} \frac{u'\iota_1}{\sqrt{n}} \sum_{i=1}^n \xi_i \left[A_i 1\{S_i = s\}\eta_{i,1}(s,\tau) + \pi 1\{S_i = s\} m_1(s,\tau)\right] + \sum_{s\in\mathcal{S}} \frac{u'\iota_2 D^w_n(s) m_1(s,\tau)}{\sqrt{n}}\\
&+ h^w_{1,1}(\tau) + R^w_{sfe,1,1}(u,\tau),
\end{aligned} \tag{S.C.1}$$

where $\eta_{i,1}(s,\tau) = (\tau - 1\{Y_i(1) \leq q_1(\tau)\}) - m_1(s,\tau)$,

$$h^w_{1,1}(\tau) = \sum_{s\in\mathcal{S}}(\pi - \hat{\pi}^w(s))q(\tau)\left(\sum_{i=1}^n \xi_i A_i 1\{S_i = s\}(\tau - 1\{Y_i \leq q_1(\tau)\})\right),$$

and

$$R^w_{sfe,1,1}(u,\tau) = -\sum_{s\in\mathcal{S}} \frac{u_1 D^w_n(s)}{\sqrt{n}n^w(s)}\left\{\sum_{i=1}^n \xi_i \left[A_i 1\{S_i = s\}\eta_{i,1}(s,\tau) + (A_i - \pi)1\{S_i = s\}m_1(s,\tau)\right]\right\}. \tag{S.C.2}$$

By Lemma S.E.2, we have

$$\sup_{\tau\in\Upsilon}|R^w_{sfe,1,1}(u,\tau)| = o_p(1).$$

Similarly, we have

$$\begin{aligned}
&L^w_{1,0,n}(u,\tau)\\
=&\sum_{s\in\mathcal{S}}\sum_{i=1}^n \xi_i\left\{\frac{u'\iota_0}{\sqrt{n}}\left[(1-A_i)1\{S_i = s\}\eta_{i,0}(s,\tau) + \pi 1\{S_i = s\}m_1(s,\tau)\right] - \frac{u'\iota_2}{\sqrt{n}}(A_i - \pi)1\{S_i = s\}m_0(s,\tau)\right\}\\
&+ h^w_{1,0}(\tau) + R^w_{sfe,1,0}(u,\tau),
\end{aligned} \tag{S.C.3}$$

where

$$\sup_{\tau\in\Upsilon}|R^w_{sfe,1,0}(u,\tau)| = o_p(1).$$

Combining (S.C.1) and (S.C.3), we have

$$\begin{aligned}
&L^w_{1,n}(u,\tau)\\
=&\frac{1}{\sqrt{n}}\sum_{s\in\mathcal{S}}\sum_{i=1}^n \xi_i\Big[u'\iota_1 A_i 1\{S_i = s\}\eta_{i,1}(u,\tau) + u'\iota_0(1-A_i)1\{S_i = s\}\eta_{i,0}(u,\tau)\\
&+ u'\iota_2(A_i - \pi)1\{S_i = s\}(m_1(s,\tau) - m_0(s,\tau)) + 1\{S_i = s\}(u'\iota_1\pi m_1(s,\tau) + u'\iota_0(1-\pi)m_0(s,\tau))\Big]\\
&+ R^w_{sfe,1,1}(u,\tau) + R^w_{sfe,1,0}(u,\tau) + h^w_{1,1}(\tau) + h^w_{1,0}(\tau).
\end{aligned}$$

Furthermore, by Lemma S.E.3, we have

$$L^w_{2,1,n}(u,\tau) = \frac{\pi f_1(q_1(\tau))}{2}(u'\iota_1)^2 - \sum_{s\in\mathcal{S}} f_1(q_1(\tau)|s)\frac{\pi D^w_n(s)u'\iota_1}{\sqrt{n}}q(\tau) + h^w_{2,1}(\tau) + R^w_{sfe,2,1}(u,\tau) \tag{S.C.4}$$

and

$$L^w_{2,0,n}(u,\tau) = \frac{(1-\pi)f_0(q_0(\tau))}{2}(u'\iota_0)^2 - \sum_{s\in\mathcal{S}} f_0(q_0(\tau)|s)\frac{(1-\pi)D^w_n(s)u'\iota_0}{\sqrt{n}}q(\tau) + h^w_{2,0}(\tau) + R^w_{sfe,2,0}(u,\tau), \tag{S.C.5}$$

where

$$h^w_{2,1}(\tau) = \sum_{s\in\mathcal{S}} \frac{\pi f_1(q_1(\tau)|s)}{2}p(s)(E^w_n(s))^2 q^2(\tau),$$

$$h^w_{2,0}(\tau) = \sum_{s\in\mathcal{S}} \frac{(1-\pi)f_0(q_0(\tau)|s)}{2}p(s)(E^w_n(s))^2 q^2(\tau),$$

$$\sup_{\tau\in\Upsilon} |R^w_{sfe,2,1}(u,\tau)| = o_p(1),$$

and

$$\sup_{\tau\in\Upsilon} |R^w_{sfe,2,0}(u,\tau)| = o_p(1).$$

Therefore,

$$\begin{aligned} L^w_{2,n}(u,\tau) =& \frac{1}{2}u'Q_{sfe}(\tau)u - \sum_{s\in\mathcal{S}} q(\tau)\left[f_1(q_1(\tau)|s)\pi u'\iota_1 + f_0(q_0(\tau)|s)(1-\pi)u'\iota_0\right]\frac{D^w_n(s)}{\sqrt{n}} \\ &+ R^w_{sfe,2,1}(u,\tau) + R^w_{sfe,2,0}(u,\tau) + h^w_{2,1}(\tau) + h^w_{2,0}(\tau). \end{aligned}$$

Combining (S.C.1), (S.C.3), (S.C.4), and (S.C.5), we have

$$L^w_{sfe,n}(u,\tau) = -u'\tilde{W}^w_{sfe,n}(\tau) + \frac{1}{2}u'Q_{sfe}u + \tilde{R}^w_{sfe,n}(u,\tau) + h^w_{sfe,n}(\tau),$$

where

$$\begin{aligned} &W^w_{sfe,n}(\tau) \\ =&\frac{1}{\sqrt{n}}\sum_{s\in\mathcal{S}}\sum_{i=1}^n \xi_i\Big[\iota_1 A_i 1\{S_i=s\}\eta_{i,1}(s,\tau) + \iota_0(1-A_i)1\{S_i=s\}\eta_{i,0}(s,\tau)\Big] \\ &+ \frac{1}{\sqrt{n}}\sum_{s\in\mathcal{S}}\sum_{i=1}^n \xi_i\Big\{\iota_2(m_1(s,\tau)-m_0(s,\tau)) + q(\tau)\Big[f_1(q_1(\tau)|s)\pi\iota_1 + f_0(q_0(\tau)|s)(1-\pi)\iota_0\Big]\Big\} \\ &\times (A_i-\pi)1\{S_i=s\} + \frac{1}{\sqrt{n}}\sum_{i=1}^n \xi_i(\iota_1\pi m_1(S_i,\tau) + \iota_0(1-\pi)m_0(S_i,\tau)), \end{aligned}$$

$$h^w_{sfe,n}(\tau) = h^w_{1,1}(\tau) + h^w_{1,0}(\tau) + h^w_{2,1}(\tau) + h^w_{2,0}(\tau),$$

and

$$\sup_{\tau\in\Upsilon} |\tilde{R}^w_{sfe,n}(u,\tau)| = o_p(1).$$

In addition, by Lemma S.E.4, $\sup_{\tau\in\Upsilon} |W^w_{sfe,n}(\tau)| = O_p(1)$. Then, by Kato (2009, Theorem 2), we have

$$\sqrt{n}(\hat{\beta}^w_{sfe}(\tau) - \tilde{\beta}(\tau)) = [Q_{sfe}(\tau)]^{-1} W^w_{sfe,n}(\tau) + R^w_{sfe,n}(\tau),$$

where

$$\sup_{\tau\in\Upsilon} ||R^w_{sfe,n}(\tau)|| = o_p(1).$$

This concludes Step 1.

Step 2. We now focus on the second element of $\hat{\beta}^w_{sfe}(\tau)$. From Step 1, we know that

$$\sqrt{n}(\hat{\beta}^w_{sfe,1}(\tau) - q(\tau)) = \frac{1}{\sqrt{n}} \sum_{s\in\mathcal{S}} \sum_{i=1}^n \xi_i \mathcal{J}_i(s,\tau) + R^w_{sfe,n,1}(\tau),$$

where

$$\begin{aligned}
\mathcal{J}_i(s,\tau) = & \left[\frac{A_i 1\{S_i = s\}\eta_{i,1}(s,\tau)}{\pi f_1(q_1(\tau))} - \frac{(1-A_i)1\{S_i = s\}\eta_{i,0}(s,\tau)}{(1-\pi) f_0(q_0(\tau))}\right] \\
& + \Bigg\{\left(\frac{1-\pi}{\pi f_1(q_1(\tau))} - \frac{\pi}{(1-\pi) f_0(q_0(\tau))}\right)(m_1(s,\tau) - m_0(s,\tau)) \\
& + q(\tau)\left[\frac{f_1(q_1(\tau)|s)}{f_1(q_1(\tau))} - \frac{f_0(q_0(\tau)|s)}{f_0(q_0(\tau))}\right]\Bigg\}(A_i - \pi)1\{S_i = s\} \\
& + \left(\frac{m_1(s,\tau)}{f_1(q_1(\tau))} - \frac{m_0(s,\tau)}{f_0(q_0(\tau))}\right) 1\{S_i = s\}
\end{aligned}$$

and

$$\sup_{\tau\in\Upsilon} |R^w_{sfe,n,1}(\tau)| = o_p(1).$$

By (S.B.14), we have

$$\sqrt{n}(\hat{\beta}_{sfe,1}(\tau) - q(\tau)) = \frac{1}{\sqrt{n}} \sum_{s\in\mathcal{S}} \sum_{i=1}^n \mathcal{J}_i(s,\tau) + R_{sfe,n,1}(\tau),$$

where

$$\sup_{\tau\in\Upsilon}|R_{sfe,n,1}(\tau)| = o_p(1).$$

Taking the difference of the above two equations, we have

$$\sqrt{n}(\hat{\beta}^w_{sfe,1}(\tau) - \hat{\beta}_{sfe,1}(\tau)) = \frac{1}{\sqrt{n}}\sum_{s\in\mathcal{S}}\sum_{i=1}^{n}(\xi_i - 1)\mathcal{J}_i(s,\tau) + R^w(\tau),$$

where

$$\sup_{\tau\in\Upsilon}|R^w(\tau)| = o_p(1).$$

Lemma S.E.5 shows that, conditionally on data,

$$\frac{1}{\sqrt{n}}\sum_{s\in\mathcal{S}}\sum_{i=1}^{n}(\xi_i - 1)\mathcal{J}_i(s,\tau) \rightsquigarrow \tilde{\mathcal{B}}_{sfe}(\tau),$$

where $\tilde{\mathcal{B}}_{sfe}(\tau)$ is a Gaussian process with covariance kernel

$$\begin{aligned}
&\tilde{\Sigma}_{sfe}(\tau_1,\tau_2)\\
=&\frac{\min(\tau_1,\tau_2) - \tau_1\tau_2 - \mathbb{E}m_1(S,\tau_1)m_1(S,\tau_2)}{\pi f_1(q_1(\tau_1))f_1(q_1(\tau_2))} + \frac{\min(\tau_1,\tau_2) - \tau_1\tau_2 - \mathbb{E}m_0(S,\tau_1)m_0(S,\tau_2)}{(1-\pi)f_0(q_0(\tau_1))f_0(q_0(\tau_2))}\\
&+ \mathbb{E}\pi(1-\pi)\Bigg[(m_1(S,\tau_1) - m_0(S,\tau_1))\left(\frac{1-\pi}{\pi f_1(q_1(\tau_1))} - \frac{\pi}{(1-\pi)f_0(q_0(\tau_1))}\right)\\
&+ q(\tau_1)\left(\frac{f_1(q(\tau_1)|S)}{f_1(q_1(\tau_1))} - \frac{f_0(q(\tau_1)|S)}{f_0(q_0(\tau_1))}\right)\Bigg]\\
&\times\left[(m_1(S,\tau_2) - m_0(S,\tau_2))\left(\frac{1-\pi}{\pi f_1(q_1(\tau_2))} - \frac{\pi}{(1-\pi)f_0(q_0(\tau_2))}\right) + q(\tau_2)\left(\frac{f_1(q(\tau_2)|S)}{f_1(q_2(\tau_2))} - \frac{f_0(q(\tau_2)|S)}{f_0(q_0(\tau_2))}\right)\right]\\
&+ \mathbb{E}\left[\frac{m_1(S,\tau_1)}{f_1(q_1(\tau_1))} - \frac{m_0(S,\tau_1)}{f_0(q_0(\tau_1))}\right]\left[\frac{m_1(S,\tau_2)}{f_1(q_1(\tau_2))} - \frac{m_0(S,\tau_2)}{f_0(q_0(\tau_2))}\right]. \qquad \text{(S.C.6)}
\end{aligned}$$

This concludes the proof for the SFE estimator.

S.D Proof of Theorem S.A.3

Recall the definition of $\tilde{\beta}(\tau) = (\tilde{\beta}_0(\tau), \tilde{\beta}_1(\tau))'$ in the proof of Theorem S.A.1. Let $u_0 = \sqrt{n}(b_0 - \tilde{\beta}_0(\tau))$, $u_1 = \sqrt{n}(b_1 - \tilde{\beta}_1(\tau))$ and $u = (u_0, u_1)' \in \Re^2$. Then,

$$\sqrt{n}(\hat{\beta}^*_{sfe}(\tau) - \tilde{\beta}(\tau)) = \arg\min_u L^*_{sfe,n}(u,\tau),$$

where

$$L^*_{sfe,n}(u,\tau) = \sum_{i=1}^{n}\left[\rho_\tau(Y_i^* - \dot{\breve{A}}_i^{*\prime}(\tilde{\beta}(\tau) + \frac{u}{\sqrt{n}})) - \rho_\tau(Y_i^* - \breve{A}_i^{*\prime}\tilde{\beta}(\tau))\right]$$

and $\breve{A}_i^* = (1, A_i^* - \pi)'$. Following the proof of Theorem S.A.1, we divide the current proof into two steps. In the first step, we show that there exist

$$g^*_{sfe,n}(u,\tau) = -u'W^*_{sfe,n}(\tau) + \frac{1}{2}u'Q_{sfe}(\tau)u$$

and $h^*_{sfe,n}(\tau)$ independent of u such that for each u

$$\sup_{\tau\in\Upsilon}|L^*_{sfe,n}(u,\tau) - g^*_{sfe,n}(u,\tau) - h^*_{sfe,n}(\tau)| \xrightarrow{p} 0.$$

In addition, we show that $\sup_{\tau\in\Upsilon}||W^*_{sfe,n}(\tau)|| = O_p(1)$. Then, by Kato (2009, Theorem 2), we have

$$\sqrt{n}(\hat{\beta}^*_{sfe}(\tau) - \tilde{\beta}(\tau)) = [Q_{sfe}(\tau)]^{-1}W^*_{sfe,n}(\tau) + R^*_{sfe,n}(\tau),$$

where

$$\sup_{\tau\in\Upsilon}||R^*_{sfe,n}(\tau)|| = o_p(1).$$

In the second step, we show that, conditionally on data,

$$\sqrt{n}(\hat{\beta}^*_{sfe,1}(\tau) - \hat{q}(\tau)) \rightsquigarrow \mathcal{B}_{sfe}(\tau).$$

Step 1. Following Step 1 in the proof of Theorem S.A.1, we have

$$L^*_{sfe,n}(u,\tau) \equiv -L^*_{1,n}(u,\tau) + L^*_{2,n}(u,\tau),$$

where

$$\begin{aligned}
&L^*_{1,n}(u,\tau)\\
=&\sum_{i=1}^{n}\sum_{s\in\mathcal{S}} A_i^* 1\{S_i^* = s\}\left(\frac{u_0}{\sqrt{n}} + (1-\hat{\pi}^*(s))\frac{u_1}{\sqrt{n}} + (\pi - \hat{\pi}^*(s))q(\tau)\right)(\tau - 1\{Y_i^* \leq q_1(\tau)\})\\
&+\sum_{i=1}^{n}\sum_{s\in\mathcal{S}}(1-A_i^*)1\{S_i^* = s\}\left(\frac{u_0}{\sqrt{n}} - \hat{\pi}^*(s)\frac{u_1}{\sqrt{n}} + (\pi - \hat{\pi}^*(s))q(\tau)\right)(\tau - 1\{Y_i^* \leq q_0(\tau)\})\\
\equiv& L^*_{1,1,n}(u,\tau) + L^*_{1,0,n}(u,\tau),
\end{aligned}$$

$L^*_{2,n}(u,\tau)$

$$
\begin{aligned}
=&\sum_{s\in\mathcal{S}}\sum_{i=1}^{n} A_i^* 1\{S_i^*=s\}\int_0^{\frac{u'\iota_1}{\sqrt{n}}-\frac{E_n^*(s)}{\sqrt{n}}\left(q(\tau)+\frac{u_1}{\sqrt{n}}\right)}\left(1\{Y_i^*\leq q_1(\tau)+v\}-1\{Y_i^*\leq q_1(\tau)\}\right)dv\\
&+\sum_{s\in\mathcal{S}}\sum_{i=1}^{n}(1-A_i^*)1\{S_i^*=s\}\int_0^{\frac{u'\iota_0}{\sqrt{n}}-\frac{E_n^*(s)}{\sqrt{n}}\left(q(\tau)+\frac{u_1}{\sqrt{n}}\right)}\left(1\{Y_i^*\leq q_0(\tau)+v\}-1\{Y_i^*\leq q_0(\tau)\}\right)dv\\
\equiv& L_{2,1,n}^*(u,\tau)+L_{2,0,n}^*(u,\tau),
\end{aligned}
$$

and $E_n^*(s)=\sqrt{n}(\hat{\pi}^*(s)-\pi)$.

Step 1.1. Recall that $\iota_1=(1,1-\pi)'$ and $\iota_0=(1,-\pi)'$. In addition, $\hat{\pi}^*(s)-\pi=\frac{D_n^*(s)}{n^*(s)}$. Then,

$$
\begin{aligned}
&L_{1,1,n}^*(u,\tau)\\
=&\sum_{s\in\mathcal{S}}\frac{u'\iota_1}{\sqrt{n}}\sum_{i=1}^{n}\left[A_i^*1\{S_i^*=s\}\eta_{i,1}^*(s,\tau)+(A_i^*-\pi)1\{S_i^*=s\}m_1(s,\tau)+\pi 1\{S_i^*=s\}m_1(s,\tau)\right]\\
&-\sum_{s\in\mathcal{S}}\frac{u_1D_n^*(s)\pi m_1(s,\tau)}{\sqrt{n}}+h_{1,1}^*(\tau)+R_{sfe,1,1}^*(u,\tau),
\end{aligned}
\tag{S.D.1}
$$

where $\eta_{i,1}^*(s,\tau)=(\tau-1\{Y_i^*(1)\leq q_1(\tau)\})-m_1(s,\tau)$,

$$
h_{1,1}^*(\tau)=\sum_{s\in\mathcal{S}}(\pi-\hat{\pi}^*(s))q(\tau)\left(\sum_{i=1}^{n}A_i^*1\{S_i^*=s\}(\tau-1\{Y_i^*\leq q_1(\tau)\})\right),
$$

and

$$
R_{sfe,1,1}^*(u,\tau)=-\sum_{s\in\mathcal{S}}\frac{u_1D_n^*(s)}{\sqrt{n}n^*(s)}\left\{\sum_{i=1}^{n}A_i^*1\{S_i^*=s\}\eta_{i,1}^*(s,\tau)+(A_i^*-\pi)1\{S_i^*=s\}m_1(s,\tau)\right\}.
\tag{S.D.2}
$$

Note that

$$
\sup_{s\in\mathcal{S},\tau\in\Upsilon}|\sum_{i=1}^{n}(A_i^*-\pi)1\{S_i^*=s\}m_1(s,\tau)|=\sup_{s\in\mathcal{S},\tau\in\Upsilon}|D_n^*(s)m_1(s,\tau)|=O_p(\sqrt{n}).
$$

In addition, Lemma E.5 shows

$$
\sup_{s\in\mathcal{S},\tau\in\Upsilon}|\sum_{i=1}^{n}A_i^*1\{S_i^*=s\}\eta_{i,1}^*(s,\tau)|=O_p(\sqrt{n(s)}).
$$

Therefore, we have

$$
\sup_{\tau\in\Upsilon}|R_{sfe,1,1}^*(u,\tau)|
$$

$$\leq \sum_{s\in\mathcal{S}} \sup_{s\in\mathcal{S}} \left|\frac{u_1 D_n^*(s)}{\sqrt{n}n^*(s)}\right| \left[\sup_{s\in\mathcal{S},\tau\in\Upsilon} |\sum_{i=1}^n A_i^* 1\{S_i^*=s\}\eta_{i,1}^*(s,\tau)| + \sup_{s\in\mathcal{S},\tau\in\Upsilon} |\sum_{i=1}^n (A_i^*-\pi)1\{S_i^*=s\}m_1(s,\tau)|\right]$$
$$=O_p(1/\sqrt{n}).$$

Similarly, we have

$$\begin{aligned}
&L_{1,0,n}^*(u,\tau)\\
=&\sum_{s\in\mathcal{S}} \frac{u'\iota_0}{\sqrt{n}} \sum_{i=1}^n \left[(1-A_i^*)1\{S_i^*=s\}\eta_{i,1}^*(s,\tau) - (A_i^*-\pi)1\{S_i^*=s\}m_0(s,\tau) + (1-\pi)1\{S_i^*=s\}m_0(s,\tau)\right]\\
&-\sum_{s\in\mathcal{S}} \frac{u_1 D_n^*(s)(1-\pi)m_0(s,\tau)}{\sqrt{n}} + h_{1,0}^*(\tau) + R_{sfe,1,0}^*(u,\tau), \qquad \text{(S.D.3)}
\end{aligned}$$

where

$$h_{1,0}^*(\tau) = \sum_{s\in\mathcal{S}} (\pi-\hat{\pi}^*(s))q(\tau)\left(\sum_{i=1}^n (1-A_i^*)1\{S_i^*=s\}(\tau - 1\{Y_i^*\leq q_0(\tau)\})\right),$$

and

$$R_{sfe,1,0}^*(u,\tau) = -\sum_{s\in\mathcal{S}} \frac{u_1 D_n^*(s)}{\sqrt{n}n^*(s)} \left\{\sum_{i=1}^n (1-A_i^*)1\{S_i^*=s\}\eta_{i,0}^*(s,\tau) - (A_i^*-\pi)1\{S_i^*=s\}m_0(s,\tau)\right\} \qquad \text{(S.D.4)}$$

such that

$$\sup_{\tau\in\Upsilon} |R_{sfe,1,0}^*(u,\tau)| = O_p(1/\sqrt{n}).$$

Therefore,

$$\begin{aligned}
L_{1,n}^*(u,\tau) =& \frac{1}{\sqrt{n}} \sum_{s\in\mathcal{S}} \sum_{i=1}^n \left[u'\iota_1 A_i^* 1\{S_i^*=s\}\eta_{i,1}^*(s,\tau) + u'\iota_0 (1-A_i^*)1\{S_i^*=s\}\eta_{i,0}^*(s,\tau)\right]\\
&+\sum_{s\in\mathcal{S}} u'\iota_2 \frac{D_n^*(s)}{\sqrt{n}} \left(m_1(s,\tau)-m_0(s,\tau)\right)\\
&+\frac{1}{\sqrt{n}} \sum_{i=1}^n (u'\iota_1 \pi m_1(S_i^*,\tau) + u'\iota_0 (1-\pi) m_0(S_i^*,\tau))\\
&+R_{sfe,1,1}^*(u,\tau) + R_{sfe,1,0}^*(u,\tau) + h_{1,1}(\tau) + h_{1,0}(\tau).
\end{aligned}$$

Furthermore, by Lemma S.E.6, we have

$$L^*_{2,1,n}(u,\tau) = \frac{\pi f_1(q_1(\tau))}{2}(u'\iota_1)^2 - \sum_{s\in\mathcal{S}} f_1(q_1(\tau)|s)\frac{\pi D^*_n(s)u'\iota_1}{\sqrt{n}}q(\tau) + h^*_{2,1}(\tau) + R^*_{sfe,2,1}(u,\tau) \quad \text{(S.D.5)}$$

and

$$L^*_{2,0,n}(u,\tau) = \frac{(1-\pi)f_0(q_0(\tau))}{2}(u'\iota_0)^2 - \sum_{s\in\mathcal{S}} f_0(q_0(\tau)|s)\frac{(1-\pi)D^*_n(s)u'\iota_0}{\sqrt{n}}q(\tau) + h^*_{2,0}(\tau) + R^*_{sfe,2,0}(u,\tau), \quad \text{(S.D.6)}$$

where

$$h^*_{2,1}(\tau) = \sum_{s\in\mathcal{S}} \frac{\pi f_1(q_1(\tau)|s)}{2}p(s)(E^*_n(s))^2 q^2(\tau),$$

$$h^*_{2,0}(\tau) = \sum_{s\in\mathcal{S}} \frac{(1-\pi) f_0(q_0(\tau)|s)}{2}p(s)(E^*_n(s))^2 q^2(\tau),$$

$$\sup_{\tau\in\Upsilon} |R^*_{sfe,2,1}(u,\tau)| = o_p(1),$$

and

$$\sup_{\tau\in\Upsilon} |R^*_{sfe,2,0}(u,\tau)| = o_p(1).$$

Therefore,

$$\begin{aligned} L^*_{2,n}(u,\tau) =& \frac{1}{2}u'Q_{sfe}(\tau)u - \sum_{s\in\mathcal{S}} q(\tau)\left[f_1(q_1(\tau)|s)\pi u'\iota_1 + f_0(q_0(\tau)|s)(1-\pi)u'\iota_0\right]\frac{D^*_n(s)}{\sqrt{n}} \\ &+ R^*_{sfe,2,1}(u,\tau) + R^*_{sfe,2,0}(u,\tau) + h^*_{2,1}(\tau) + h^*_{2,0}(\tau). \end{aligned}$$

Combining (S.D.1), (S.D.3), (S.D.5), and (S.D.6), we have

$$L^*_{sfe,n}(u,\tau) = -u'W^*_{sfe,n}(\tau) + \frac{1}{2}u'Q_{sfe}u + \tilde{R}^*_{sfe,n}(u,\tau) + h^*_{sfe,n}(\tau),$$

where

$$\begin{aligned} &W^*_{sfe,n}(\tau) \\ =&\frac{1}{\sqrt{n}}\sum_{s\in\mathcal{S}}\sum_{i=1}^{n}\left[\iota_1 A^*_i 1\{S^*_i = s\}\eta^*_{i,1}(s,\tau) + \iota_0(1-A^*_i)1\{S^*_i = s\}\eta^*_{i,0}(s,\tau)\right] \end{aligned}$$

$$+\sum_{s\in\mathcal{S}}\left\{\iota_2(m_1(s,\tau)-m_0(s,\tau))+q(\tau)\left[f_1(q_1(\tau)|s)\pi\iota_1+f_0(q_0(\tau)|s)(1-\pi)\iota_0\right]\right\}\frac{D_n^*(s)}{\sqrt{n}}$$
$$+\frac{1}{\sqrt{n}}\sum_{i=1}^n(\iota_1\pi m_1(S_i^*,\tau)+\iota_0(1-\pi)m_0(S_i^*,\tau)),$$

$$h_{sfe,n}^*(\tau)=h_{1,1}^*(\tau)+h_{1,0}^*(\tau)+h_{2,1}^*(\tau)+h_{2,0}^*(\tau),$$

and

$$\sup_{\tau\in\Upsilon}|\tilde{R}_{sfe,n}^*(u,\tau)|=o_p(1).$$

By Lemma S.E.7, $\sup_{\tau\in\Upsilon}|W_{sfe,n}^*(\tau)|=O_p(1)$. Then, by Kato (2009, Theorem 2), we have

$$\sqrt{n}(\hat{\beta}_{sfe}^*(\tau)-\tilde{\beta}(\tau))=[Q_{sfe}(\tau)]^{-1}W_{sfe,n}^*(\tau)+R_{sfe,n}^*(\tau),$$

where

$$\sup_{\tau\in\Upsilon}||R_{sfe,n}^*(\tau)||=o_p(1).$$

This concludes Step 1.

Step 2. We now focus on the second element of $\hat{\beta}_{sfe}^*(\tau)$. From Step 1, we know that

$$\begin{aligned}
&\sqrt{n}(\hat{\beta}_{sfe,1}^*(\tau)-q(\tau))\\
=&\frac{1}{\sqrt{n}}\sum_{s\in\mathcal{S}}\sum_{i=1}^n\left[\frac{A_i^*1\{S_i^*=s\}\eta_{i,1}^*(s,\tau)}{\pi f_1(q_1(\tau))}-\frac{(1-A_i^*)1\{S_i^*=s\}\eta_{i,0}^*(s,\tau)}{(1-\pi)f_0(q_0(\tau))}\right]\\
&+\sum_{s\in\mathcal{S}}\left\{\left(\frac{1-\pi}{\pi f_1(q_1(\tau))}-\frac{\pi}{(1-\pi)f_0(q_0(\tau))}\right)(m_1(s,\tau)-m_0(s,\tau))+q(\tau)\left[\frac{f_1(q_1(\tau)|s)}{f_1(q_1(\tau))}-\frac{f_0(q_0(\tau)|s)}{f_0(q_0(\tau))}\right]\right\}\frac{D_n^*(s)}{\sqrt{n}}\\
&+\frac{1}{\sqrt{n}}\sum_{i=1}^n\left(\frac{m_1(S_i^*,\tau)}{f_1(q_1(\tau))}-\frac{m_0(S_i^*,\tau)}{f_0(q_0(\tau))}\right)+R_{sfe,n,1}^*(\tau)\\
\equiv&W_{sfe,n,1}^*(\tau)+W_{sfe,n,2}^*(\tau)+W_{sfe,n,3}^*(\tau)+R_{sfe,n,1}^*(\tau),
\end{aligned}$$

where

$$\sup_{\tau\in\Upsilon}|R_{sfe,n,1}^*(\tau)|=o_p(1).$$

By (B.4), we have

$$\sqrt{n}(\hat{q}(\tau)-q(\tau))$$

$$
\begin{aligned}
=&\frac{1}{\sqrt{n}}\sum_{s\in\mathcal{S}}\sum_{i=1}^{n}\left[\frac{A_i 1\{S_i=s\}\eta_{i,1}(s,\tau)}{\pi f_1(q_1(\tau))}-\frac{(1-A_i)1\{S_i=s\}\eta_{i,0}(s,\tau)}{(1-\pi)f_0(q_0(\tau))}\right]\\
&+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{m_1(S_i,\tau)}{f_1(q_1(\tau))}-\frac{m_0(S_i,\tau)}{f_0(q_0(\tau))}\right)+R_{ipw,n}(\tau)\\
\equiv&\mathcal{W}_{n,1}(\tau)+\mathcal{W}_{n,2}(\tau)+R_{ipw,n}(\tau),
\end{aligned}
$$

where

$$\sup_{\tau\in\Upsilon}|R_{ipw,n}(\tau)|=o_p(1).$$

Taking the difference of the above two equations, we have

$$\sqrt{n}(\hat{\beta}^*_{sfe,1}(\tau)-\hat{q}(\tau))=(W^*_{sfe,n,1}(\tau)-\mathcal{W}_{n,1}(\tau))+W^*_{sfe,n,2}(\tau)+(W^*_{sfe,n,3}(\tau)-\mathcal{W}_{n,2}(\tau))+R^*(\tau), \tag{S.D.7}$$

where

$$\sup_{\tau\in\Upsilon}|R^*(\tau)|=o_p(1).$$

Lemma S.E.7 shows that, conditionally on data,

$$(W^*_{sfe,n,1}(\tau)-\mathcal{W}_{n,1}(\tau)),W^*_{sfe,n,2}(\tau),(W^*_{sfe,n,3}(\tau)-\mathcal{W}_{n,2}(\tau))\rightsquigarrow(\mathcal{B}_1(\tau),\mathcal{B}_2(\tau),\mathcal{B}_3(\tau)),$$

where $(\mathcal{B}_1(\tau),\mathcal{B}_2(\tau),\mathcal{B}_3(\tau))$ are three independent Gaussian processes and $\sum_{j=1}^{3}\mathcal{B}_j(\tau)\stackrel{d}{=}\mathcal{B}_{sfe}(\tau)$. This concludes the proof.

S.E Technical Lemmas

Lemma S.E.1. *Let $W_{sfe,n,j}(\tau)$, $j=1,2,3$ be defined as in (S.B.14). If Assumptions in Theorem S.A.1 hold, then uniformly over $\tau\in\Upsilon$,*

$$(W_{sfe,n,1}(\tau),W_{sfe,n,2}(\tau),W_{sfe,n,3}(\tau))\rightsquigarrow(\mathcal{B}_{sfe,1}(\tau),\mathcal{B}_{sfe,2}(\tau),\mathcal{B}_{sfe,3}(\tau)),$$

where $(\mathcal{B}_{sfe,1}(\tau),\mathcal{B}_{sfe,2}(\tau),\mathcal{B}_{sfe,3}(\tau))$ are three independent two-dimensional Gaussian process with covariance kernels $\Sigma_{sfe,1}(\tau_1,\tau_2)$, $\Sigma_{sfe,2}(\tau_1,\tau_2)$, and $\Sigma_{sfe,3}(\tau_1,\tau_2)$, respectively. The expressions for the three kernels are derived in the proof below.

Proof. The proofs of weak convergence and the independence among $(\mathcal{B}_{sfe,1}(\tau),\mathcal{B}_{sfe,2}(\tau),\mathcal{B}_{sfe,3}(\tau))$ are similar to that in Lemma E.2, and thus, are omitted. In the following, we focus on deriving the covariance kernels.

First, similar to the argument in the proof of Lemma E.2,

$$W_{sfe,n,1}(\tau) \stackrel{d}{=} \iota_1 \sum_{s\in\mathcal{S}} \sum_{i=N(s)+1}^{N(s)+n_1(s)} \frac{1}{\sqrt{n}}\tilde{\eta}_{i,1}(s,\tau) + \iota_0 \sum_{s\in\mathcal{S}} \sum_{i=N(s)+n_1(s)+1}^{N(s)+n(s)} \frac{1}{\sqrt{n}}\tilde{\eta}_{i,0}(s,\tau).$$

Therefore,

$$\begin{aligned}\Sigma_1(\tau_1,\tau_2) =&\pi[\min(\tau_1,\tau_2) - \tau_1\tau_2 - \mathbb{E}m_1(S,\tau_1)m_1(S,\tau_2)]\iota_1\iota_1' \\ &+ (1-\pi)[\min(\tau_1,\tau_2) - \tau_1\tau_2 - \mathbb{E}m_0(S,\tau_1)m_0(S,\tau_2)]\iota_0\iota_0'.\end{aligned}$$

For $W_{sfe,n,2}(\tau)$, we have

$$\begin{aligned}\Sigma_2(\tau_1,\tau_2) =&\mathbb{E}\gamma(S)\left[\iota_2(m_1(S,\tau_1)-m_0(S,\tau_1)) + q(\tau_1)\left(f_1(q_1(\tau_1)|S)\pi\iota_1 + f_0(q_0(\tau_1)|S)(1-\pi)\iota_0\right)\right] \\ &\times\left[\iota_2(m_1(S,\tau_2)-m_0(S,\tau_2)) + q(\tau_2)\left(f_1(q_1(\tau_2)|S)\pi\iota_1 + f_0(q_0(\tau_2)|S)(1-\pi)\iota_0\right)\right]'.\end{aligned}$$

Next, we have

$$\Sigma_3(\tau_1,\tau_2) = \mathbb{E}(\iota_1\pi m_1(S,\tau_1) + \iota_0(1-\pi)m_0(S,\tau_1))(\iota_1\pi m_1(S,\tau_2) + \iota_0(1-\pi)m_0(S,\tau_2))'.$$

In addition,

$$[Q_{sfe}(\tau)]^{-1} = \begin{pmatrix} \frac{1-\pi}{f_0(q_0(\tau))} + \frac{\pi}{f_1(q_1(\tau))} & \frac{1}{f_1(q_1(\tau))} - \frac{1}{f_0(q_0(\tau))} \\ \frac{1}{f_1(q_1(\tau))} - \frac{1}{f_0(q_0(\tau))} & \frac{1}{(1-\pi)f_0(q_0(\tau))} + \frac{1}{\pi f_1(q_1(\tau))} \end{pmatrix}.$$

Therefore,

$$\begin{aligned}&\Sigma(\tau_1,\tau_2) \\ =&\left\{\frac{1}{\pi f_1(q_1(\tau_1))f_1(q_1(\tau_2))}[\min(\tau_1,\tau_2) - \tau_1\tau_2 - \mathbb{E}m_1(S,\tau_1)m_1(S,\tau_2)]\begin{pmatrix}\pi^2 & \pi \\ \pi & 1\end{pmatrix}\right. \\ &\left. + \frac{1}{(1-\pi)f_0(q_0(\tau_1))f_0(q_0(\tau_2))}[\min(\tau_1,\tau_2) - \tau_1\tau_2 - \mathbb{E}m_0(S,\tau_1)m_0(S,\tau_2)]\begin{pmatrix}(1-\pi)^2 & \pi-1 \\ \pi-1 & 1\end{pmatrix}\right\} \\ &+\left\{\mathbb{E}\gamma(S)\left[(m_1(S,\tau_1)-m_0(S,\tau_1))\begin{pmatrix}\frac{\pi}{f_0(q_0(\tau_1))} + \frac{1-\pi}{f_1(q_1(\tau_1))} \\ \frac{1-\pi}{\pi f_1(q_1(\tau_1))} - \frac{\pi}{(1-\pi)f_0(q_0(\tau_1))}\end{pmatrix} + q(\tau_1)\frac{f_1(q_1(\tau_1)|S)}{f_1(q_1(\tau_1))}\begin{pmatrix}\pi \\ 1\end{pmatrix}\right.\right. \\ &\left. + q(\tau_1)\frac{f_0(q_0(\tau_1)|S)}{f_0(q_0(\tau_1))}\begin{pmatrix}1-\pi \\ -1\end{pmatrix}\right] \times \left[(m_1(S,\tau_2)-m_0(S,\tau_2))\begin{pmatrix}\frac{\pi}{f_0(q_0(\tau_2))} + \frac{1-\pi}{f_1(q_1(\tau_2))} \\ \frac{1-\pi}{\pi f_1(q_1(\tau_2))} - \frac{\pi}{(1-\pi)f_0(q_0(\tau_2))}\end{pmatrix}\right. \\ &\left.\left. + q(\tau_2)\frac{f_1(q_1(\tau_2)|S)}{f_1(q_1(\tau_2))}\begin{pmatrix}\pi \\ 1\end{pmatrix} + q(\tau_2)\frac{f_0(q_0(\tau_2)|S)}{f_0(q_0(\tau_2))}\begin{pmatrix}1-\pi \\ -1\end{pmatrix}\right]\right\}\end{aligned}$$

$$+\left\{\mathbb{E}\left[\frac{m_1(S,\tau_1)}{f_1(q_1(\tau_1))}\begin{pmatrix}\pi\\1\end{pmatrix}+\frac{m_0(S,\tau_1)}{f_0(q_0(\tau_1))}\begin{pmatrix}1-\pi\\-1\end{pmatrix}\right]\left[\frac{m_1(S,\tau_2)}{f_1(q_1(\tau_2))}\begin{pmatrix}\pi\\1\end{pmatrix}+\frac{m_0(S,\tau_2)}{f_0(q_0(\tau_2))}\begin{pmatrix}1-\pi\\-1\end{pmatrix}\right]'\right\}.$$

□

Lemma S.E.2. *Recall the definition of $R^w_{sfe,1,1}(u,\tau)$ in (S.C.2). If Assumptions 1 and 2 hold, then*

$$\sup_{\tau\in\Upsilon}|R^w_{sfe,1,1}(u,\tau)| = o_p(1).$$

Proof. We divide the proof into two steps. In the first step, we show that $\sup_{s\in\mathcal{S}}|D^w_n(s)| = O_p(\sqrt{n})$. In the second step, we show that

$$\sup_{\tau\in\Upsilon,s\in\mathcal{S}}|\sum_{i=1}^n \xi_i A_i 1\{S_i=s\}\eta_{i,1}(s,\tau)| = O_p(\sqrt{n}). \tag{S.E.1}$$

Then,

$$\begin{aligned}
&\sup_{\tau\in\Upsilon}|R^w_{sfe,1,1}(u,\tau)|\\
\leq&\sum_{s\in\mathcal{S}}\frac{|u_1|}{n^w(s)}\sup_{s\in\mathcal{S}}\left|\frac{D^w_n(s)}{\sqrt{n}}\right|\left[\sup_{\tau\in\Upsilon,s\in\mathcal{S}}\left|\sum_{i=1}^n\xi_i A_i 1\{S_i=s\}\eta_{i,1}(s,\tau)\right|+\sup_{s\in\mathcal{S}}|D^w_n(s)|\right]\\
=&O_p(1/\sqrt{n}),
\end{aligned}$$

as $n^w(s)/n \xrightarrow{p} p(s)>0$.

Step 1. Because

$$\sup_{s\in\mathcal{S}}|D_n(s)| = O_p(\sqrt{n}),$$

we only need to bound the difference $D^w_n(s)-D_n(s)$. Note that

$$n(s)^{-1/2}D^w_n(s)-n(s)^{-1/2}D_n(s) = n^{-1/2}\sum_{i=1}^n(\xi_i-1)(A_i-\pi)1\{S_i=s\}. \tag{S.E.2}$$

We aim to prove that, if $n(s)\to\infty$ and $D_n(s)/n(s)=o_p(1)$, then conditionally on data, for $s\in\mathcal{S}$,

$$n(s)^{-1/2}\sum_{i=1}^n(\xi_i-1)(A_i-\pi)1\{S_i=s\} \rightsquigarrow N(0,\pi(1-\pi)) \tag{S.E.3}$$

and they are independent across $s \in \mathcal{S}$. The independence is straightforward because

$$\frac{1}{n(s)}\sum_{i=1}^{n}(\xi_i-1)^2(A_i-\pi)^2 1\{S_i=s\}1\{S_i=s'\}=0 \quad \text{for} \quad s\neq s'.$$

For the limiting distribution, let $\mathcal{D}_n=\{Y_i,A_i,S_i\}_{i=1}^n$ denote data. According to the Lindeberg-Feller central limit theorem, (S.E.3) holds because (1)

$$\begin{aligned}
n(s)^{-1}\sum_{i=1}^{n}\mathbb{E}[(\xi_i-1)^2(A_i-\pi)^2 1\{S_i=s\}|\mathcal{D}_n] =& n(s)^{-1}\sum_{i=1}^{n}(A_i-2A_i\pi+\pi^2)1\{S_i=s\}\\
=& n(s)^{-1}\sum_{i=1}^{n}(A_i-\pi-2(A_i-\pi)\pi+\pi-\pi^2)1\{S_i=s\}\\
=& \frac{1-2\pi}{n(s)}D_n(s)+\pi(1-\pi)\\
\xrightarrow{p}& \pi(1-\pi),
\end{aligned}$$

and (2) for every $\varepsilon>0$,

$$\begin{aligned}
& n(s)^{-1}\sum_{i=1}^{n}(A_i-\pi)^2 1\{S_i=s\}\mathbb{E}\left[(\xi_i-1)^2 1\{|\xi_i-1|(A_i-\pi)^2 1\{S_i=s\}>\varepsilon\sqrt{n(s)}\}|\mathcal{D}_n\right]\\
\leq & 4\mathbb{E}(\xi_i-1)^2 1\{2|\xi_i-1|\geq\varepsilon\sqrt{n(s)}\}\rightarrow 0,
\end{aligned}$$

where we use the fact that $|A_i-\pi|1\{S_i=s\}\leq 2$ and $n(s)\rightarrow\infty$. This concludes the proof of Step 1.

Step 2. By the same rearrangement argument and the fact that $\{\xi_i\}_{i=1}^n \perp\!\!\!\perp \mathcal{D}_n$, we have

$$\sup_{\tau\in\Upsilon,s\in\mathcal{S}}\left|\frac{1}{n}\sum_{i=1}^{n}\xi_i A_i 1\{S_i=s\}\eta_{i,1}(s,\tau)\right| \stackrel{d}{=} \sup_{\tau\in\Upsilon,s\in\mathcal{S}}\left|\frac{1}{n}\sum_{i=N(s)+1}^{N(s)+n_1(s)}\xi_i\tilde{\eta}_{i,1}(s,\tau)\right|.$$

Let $\Gamma_{n,1}(s,t,\tau)=\sum_{i=1}^{\lfloor nt\rfloor}\frac{\xi_i\tilde{\eta}_{i,1}(s,\tau)}{\sqrt{n}}$ and $\mathcal{F}=\{\xi_i\tilde{\eta}_{i,1}(s,\tau):\tau\in\Upsilon,s\in\mathcal{S}\}$ with envelope $F_i=C\xi_i$ and $||F_i||_{P,2}<\infty$. By Lemma E.1 and van der Vaart and Wellner (1996, Theorem 2.14.1), for any $\varepsilon>0$, we can choose M sufficiently large such that

$$\begin{aligned}
\mathbb{P}(\sup_{0<t\leq 1,\tau\in\Upsilon,s\in\mathcal{S}}|\Gamma_{n,1}(s,t,\tau)|\geq M)\leq & \frac{270\mathbb{E}\sup_{\tau\in\Upsilon,s\in\mathcal{S}}|\Gamma_{n,1}(s,1,\tau)|}{M}\\
= & \frac{270\mathbb{E}\sqrt{n}||\mathbb{P}_n-\mathbb{P}||_{\mathcal{F}}}{M}\lesssim\frac{J(1,\mathcal{F})||F_i||_{P,2}}{M}<\varepsilon.
\end{aligned}$$

Therefore,

$$\sup_{0<t\leq 1,\tau\in\Upsilon,s\in\mathcal{S}}|\Gamma_{n,1}(s,t,\tau)|=O_p(1)$$

and

$$\sup_{\tau\in\Upsilon,s\in\mathcal{S}}\left|\frac{1}{n}\sum_{i=1}^{n}\xi_iA_i1\{S_i=s\}\eta_{i,1}(s,\tau)\right| \stackrel{d}{=} \sup_{\tau\in\Upsilon,s\in\mathcal{S}}\frac{1}{\sqrt{n}}\left|\Gamma_{n,1}\left(s,\frac{N(s)+n_1(s)}{n},\tau\right)-\Gamma_{n,1}\left(s,\frac{N(s)}{n},\tau\right)\right|$$
$$=O_p(1/\sqrt{n}). \tag{S.E.4}$$

This concludes the proof of Step 2. ☐

Lemma S.E.3. *If Assumptions 1 and 2 hold, then S.C.4 and S.C.5 hold.*

Proof. We focus on (S.C.4). Note that

$$\begin{aligned}
&L^w_{2,1,n}(u,\tau)\\
=&\sum_{s\in\mathcal{S}}\sum_{i=1}^{n}\xi_iA_i1\{S_i=s\}\int_0^{\frac{u'\iota_1}{\sqrt{n}}-\frac{E_n^w(s)}{\sqrt{n}}\left(q(\tau)+\frac{u_1}{\sqrt{n}}\right)}\left(1\{Y_i(1)\le q_1(\tau)+v\}-1\{Y_i(1)\le q_1(\tau)\}\right)dv\\
=&\sum_{s\in\mathcal{S}}\sum_{i=1}^{n}\xi_iA_i1\{S_i=s\}[\phi_i(u,\tau,s,E_n^w(s))-\mathbb{E}\phi_i(u,\tau,s,E_n^w(s)|S_i=s)]\\
&+\sum_{s\in\mathcal{S}}\sum_{i=1}^{n}\xi_iA_i1\{S_i=s\}\mathbb{E}\phi_i(u,\tau,s,E_n^w(s)|S_i=s),
\end{aligned} \tag{S.E.5}$$

where by Lemma S.E.2, $E_n^w(s)=\sqrt{n}(\hat{\pi}^w(s)-\pi)=\frac{n}{n^w(s)}\frac{D_n^w(s)}{\sqrt{n}}=O_p(1)$,

$$\phi_i(u,\tau,s,e)=\int_0^{\frac{u'\iota_1}{\sqrt{n}}-\frac{e}{\sqrt{n}}\left(q(\tau)+\frac{u_1}{\sqrt{n}}\right)}\left(1\{Y_i(1)\le q_1(\tau)+v\}-1\{Y_i(1)\le q_1(\tau)\}\right)dv,$$

and $\mathbb{E}\phi_i(u,\tau,s,E_n^w(s)|S_i=s)$ is interpreted as $\mathbb{E}(\phi_i(u,\tau,s,e)|S_i=s)$ with e being evaluated at $E_n^w(s)$.

For the first term on the RHS of (S.E.5), by the rearrangement argument in Lemma E.2, we have

$$\begin{aligned}
&\sum_{s\in\mathcal{S}}\sum_{i=1}^{n}\xi_iA_i1\{S_i=s\}[\phi_i(u,\tau,s,E_n^w(s))-\mathbb{E}\phi_i(u,\tau,s,E_n^w(s)|S_i=s)]\\
\stackrel{d}{=}&\sum_{s\in\mathcal{S}}\sum_{i=N(s)+1}^{N(s)+n_1(s)}\xi_i[\phi_i^s(u,\tau,s,E_n^w(s))-\mathbb{E}\phi_i^s(u,\tau,s,E_n^w(s))],
\end{aligned}$$

where

$$\phi_i^s(u,\tau,s,e)=\int_0^{\frac{u'\iota_1}{\sqrt{n}}-\frac{e}{\sqrt{n}}\left(q(\tau)+\frac{u_1}{\sqrt{n}}\right)}\left(1\{Y_i^s(1)\le q_1(\tau)+v\}-1\{Y_i^s(1)\le q_1(\tau)\}\right)dv.$$

Similar to (S.B.9), we can show that, as $n \to \infty$,

$$\sup_{\tau \in \Upsilon, s \in \mathcal{S}} \left| \sum_{i=N(s)+1}^{N(s)+n_1(s)} \xi_i \left[\phi_i^s(u,\tau,s,E_n^w(s)) - \mathbb{E}\phi_i^s(u,\tau,s,E_n^w(s)) \right] \right| = o_p(1). \tag{S.E.6}$$

For the second term in (S.E.5), we have

$$\begin{aligned}
&\sum_{s\in\mathcal{S}} \sum_{i=1}^n \xi_i A_i 1\{S_i = s\} \mathbb{E}\phi_i(u,\tau,s,E_n^w(s)|S_i = s) \\
=&\sum_{s\in\mathcal{S}} \frac{\sum_{i=1}^n \xi_i \pi 1\{S_i = s\}}{n} n\mathbb{E}\phi_i^s(u,\tau,s,E_n^w(s)) + \sum_{s\in\mathcal{S}} \frac{D_n^w(s)}{n} n\mathbb{E}\phi_i^s(u,\tau,s,E_n^w(s)) \\
=&\sum_{s\in\mathcal{S}} \pi p(s) \left[\frac{f_1(q_1(\tau)|s)}{2} (u'\iota_1 - E_n^w(s)q(\tau))^2 + o_p(1) \right] + \sum_{s\in\mathcal{S}} \frac{D_n^w(s)}{n} \left[\frac{f_1(q_1(\tau)|s)}{2} (u'\iota_1 - E_n^w(s)q(\tau))^2 + o_p(1) \right] \\
=&\frac{\pi f_1(q_1(\tau))}{2}(u'\iota_1)^2 - \sum_{s\in\mathcal{S}} f_1(q_1(\tau)|s) \frac{\pi D_n^w(s) u'\iota_1}{\sqrt{n}} q(\tau) + h_{2,1}^w(\tau) + o_p(1),
\end{aligned} \tag{S.E.7}$$

where the $o_p(1)$ term holds uniformly over $(\tau, s) \in \Upsilon \times \mathcal{S}$. The second equality holds by the same calculation in (S.B.10) and the fact that $\sum_{i=1}^n \xi_i 1\{S_i = s\}/n \xrightarrow{p} p(s)$. The last inequality holds because $\frac{D_n^w(s)}{n} = o_p(1)$, $E_n^w(s) = \frac{n}{n^w(s)} \frac{D_n^w(s)}{\sqrt{n}} = O_p(1)$, $\frac{n}{n^w(s)} \xrightarrow{p} 1/p(s)$, and

$$h_{2,1}^w(\tau) = \sum_{s\in\mathcal{S}} \frac{\pi f_1(q_1(\tau)|s)}{2} p(s) (E_n^w(s))^2 q^2(\tau).$$

Combining (S.E.5)–(S.E.7), we have

$$L_{2,1,n}^w(u,\tau) = \frac{\pi f_1(q_1(\tau))}{2}(u'\iota_1)^2 - \sum_{s\in\mathcal{S}} f_1(q_1(\tau)|s) \frac{\pi D_n^w(s) u'\iota_1}{\sqrt{n}} q(\tau) + h_{2,1}^w(\tau) + R_{sfe,2,1}^w(u,\tau),$$

where

$$h_{2,1}^w(\tau) = \sum_{s\in\mathcal{S}} \frac{\pi f_1(q_1(\tau)|s)}{2} p(s) (E_n^w(s))^2 q^2(\tau)$$

and

$$\sup_{\tau\in\Upsilon} |R_{sfe,2,1}^w(u,\tau)| = o_p(1).$$

This concludes the proof.

$\square$

Lemma S.E.4. *If Assumptions 1 and 2 hold, then* $\sup_{\tau\in\Upsilon} ||W_{sfe,n}^w(\tau)|| = O_p(1)$.

Proof. It suffices to show that

$$\sup_{\tau\in\Upsilon,s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_iA_i1\{S_i=s\}\eta_{i,1}(s,\tau)\right|=O_p(1) \tag{S.E.8}$$

$$\sup_{\tau\in\Upsilon,s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_i(1-A_i)1\{S_i=s\}\eta_{i,0}(s,\tau)\right|=O_p(1), \tag{S.E.9}$$

$$\sup_{s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_i(A_i-\pi)1\{S_i=s\}\right|=O_p(1), \tag{S.E.10}$$

and

$$\sup_{\tau\in\Upsilon}\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_i(\iota_1\pi m_1(S_i,\tau)+\iota_0(1-\pi)m_0(S_i,\tau))\right\|=O_p(1). \tag{S.E.11}$$

Note that (S.E.8) holds by the argument in step 2 in the proof of Lemma S.E.2, (S.E.9) holds similarly, (S.E.10) holds by (S.E.2) and (S.E.3), and (S.E.11) holds by the usual maximal inequality, e.g., van der Vaart and Wellner (1996, Theorem 2.14.1). This concludes the proof. □

Lemma S.E.5. *If Assumptions 1 and 2 hold, then conditionally on data,*

$$\frac{1}{\sqrt{n}}\sum_{s\in\mathcal{S}}\sum_{i=1}^{n}(\xi_i-1)\mathcal{J}_i(s,\tau)\rightsquigarrow\tilde{\mathcal{B}}_{sfe}(\tau),$$

where $\tilde{\mathcal{B}}_{sfe}(\tau)$ is a Gaussian process with covariance kernel $\tilde{\Sigma}_{sfe}(\cdot,\cdot)$ defined in (S.C.6).

Proof. In order to show the weak convergence, we only need to show (1) conditional stochastic equicontinuity and (2) conditional convergence in finite dimension. We divide the proof into two steps accordingly.

Step 1. In order to show the conditional stochastic equicontinuity, it suffices to show that, for any $\varepsilon>0$, as $n\to\infty$ followed by $\delta\to 0$,

$$\mathbb{P}_\xi\left(\sup_{\tau_1,\tau_2\in\Upsilon,\tau_1<\tau_2<\tau_1+\delta,s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\xi_i-1)(\mathcal{J}_i(s,\tau_2)-\mathcal{J}_i(s,\tau_1))\right|\geq\varepsilon\right)\xrightarrow{p}0,$$

where $\mathbb{P}_\xi(\cdot)$ means that the probability operator is with respect to $\xi_1,\cdots,\xi_n$ and conditional on data. Note

$$\mathbb{E}\mathbb{P}_\xi\left(\sup_{\tau_1,\tau_2\in\Upsilon,\tau_1<\tau_2<\tau_1+\delta,s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\xi_i-1)(\mathcal{J}_i(s,\tau_1)-\mathcal{J}_i(s,\tau_1))\right|\geq\varepsilon\right)$$

$$
\begin{aligned}
=&\mathbb{P}\left(\sup_{\tau_1,\tau_2\in\Upsilon,\tau_1<\tau_2<\tau_1+\delta,s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi_i-1)(\mathcal{J}_i(s,\tau_2)-\mathcal{J}_i(s,\tau_1))\right|\geq\varepsilon\right)\\
\leq&\mathbb{P}\left(\sup_{\tau_1,\tau_2\in\Upsilon,\tau_1<\tau_2<\tau_1+\delta,s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi_i-1)(\mathcal{J}_{i,1}(s,\tau_2)-\mathcal{J}_{i,1}(s,\tau_1))\right|\geq\varepsilon/3\right)\\
&+\mathbb{P}\left(\sup_{\tau_1,\tau_2\in\Upsilon,\tau_1<\tau_2<\tau_1+\delta,s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi_i-1)(\mathcal{J}_{i,2}(s,\tau_2)-\mathcal{J}_{i,2}(s,\tau_1))\right|\geq\varepsilon/3\right)\\
&+\mathbb{P}\left(\sup_{\tau_1,\tau_2\in\Upsilon,\tau_1<\tau_2<\tau_1+\delta,s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi_i-1)(\mathcal{J}_{i,3}(s,\tau_2)-\mathcal{J}_{i,3}(s,\tau_1))\right|\geq\varepsilon/3\right),
\end{aligned}
$$

where

$$
\mathcal{J}_{i,1}(s,\tau)=\frac{A_i1\{S_i=s\}\eta_{i,1}(s,\tau)}{\pi f_1(q_1(\tau))}-\frac{(1-A_i)1\{S_i=s\}\eta_{i,0}(s,\tau)}{(1-\pi)f_0(q_0(\tau))},
$$

$$
\mathcal{J}_{i,2}(s,\tau)=F_1(s,\tau)(A_i-\pi)1\{S_i=s\},
$$

$$
F_1(s,\tau)=\left(\frac{1-\pi}{\pi f_1(q_1(\tau))}-\frac{\pi}{(1-\pi)f_0(q_0(\tau))}\right)(m_1(s,\tau)-m_0(s,\tau))+q(\tau)\left[\frac{f_1(q_1(\tau)|s)}{f_1(q_1(\tau))}-\frac{f_0(q_0(\tau)|s)}{f_0(q_0(\tau))}\right],
$$

$$
\mathcal{J}_{i,3}(s,\tau)=\left(\frac{m_1(s,\tau)}{f_1(q_1(\tau))}-\frac{m_0(s,\tau)}{f_0(q_0(\tau))}\right)1\{S_i=s\}.
$$

Further note that

$$
\sum_{i=1}^n(\xi_i-1)\mathcal{J}_{i,1}(s,\tau)\stackrel{d}{=}\sum_{i=N(s)+1}^{N(s)+n_1(s)}\frac{(\xi_i-1)\tilde{\eta}_{i,1}(s,\tau)}{\pi f_1(q_1(\tau))}-\sum_{i=N(s)+n_1(s)+1}^{N(s)+n(s)}\frac{(\xi_i-1)\tilde{\eta}_{i,0}(s,\tau)}{(1-\pi)f_0(q_0(\tau))}
$$

By the same argument in Claim (1) in the proof of Lemma E.2, we have

$$
\begin{aligned}
&\mathbb{P}\left(\sup_{\tau_1,\tau_2\in\Upsilon,\tau_1<\tau_2<\tau_1+\delta,s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi_i-1)(\mathcal{J}_{i,1}(s,\tau_2)-\mathcal{J}_i(s,\tau_1))\right|\geq\varepsilon/3\right)\\
\leq&\frac{3\mathbb{E}\sup_{\tau_1,\tau_2\in\Upsilon,\tau_1<\tau_2<\tau_1+\delta,s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi_i-1)(\mathcal{J}_{i,1}(s,\tau_2)-\mathcal{J}_{i,1}(s,\tau_1))\right|}{\varepsilon}\\
\leq&\frac{3\sqrt{c_2\delta\log(\frac{C}{c_1\delta})}+\frac{3C\log(\frac{C}{c_1\delta})}{\sqrt{n}}}{\varepsilon},
\end{aligned}
$$

where C, $c_1<c_2$ are some positive constants that are independent of (n,ε,δ). By letting $n\to\infty$ followed by $\delta\to 0$, the RHS vanishes.

For $\mathcal{J}_{i,2}$, we note that $F_1(s,\tau)$ is Lipschitz in τ. Therefore,

$$\mathbb{P}\left(\sup_{\tau_1,\tau_2\in\Upsilon,\tau_1<\tau_2<\tau_1+\delta,s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi_i-1)(\mathcal{J}_{i,2}(s,\tau_2)-\mathcal{J}_{i,2}(s,\tau_1))\right|\geq\varepsilon/3\right)$$
$$\leq\sum_{s\in\mathcal{S}}\mathbb{P}\left(C\delta\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi_i-1)(A_i-\pi)1\{S_i=s\}\right|\geq\varepsilon/3\right)\to 0$$

as $n\to\infty$ followed by $\delta\to 0$, in which we use the fact that, by (S.E.3),

$$\sup_{s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi_i-1)(A_i-\pi)1\{S_i=s\}\right|=O_p(1).$$

Last, by the standard maximal inequality (e.g., van der Vaart and Wellner (1996, Theorem 2.14.1)) and the fact that

$$\left(\frac{m_1(s,\tau)}{f_1(q_1(\tau))}-\frac{m_0(s,\tau)}{f_0(q_0(\tau))}\right)$$

is Lipschitz in τ, we have, as $n\to\infty$ followed by $\delta\to 0$,

$$\mathbb{P}\left(\sup_{\tau_1,\tau_2\in\Upsilon,\tau_1<\tau_2<\tau_1+\delta,s\in\mathcal{S}}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi_i-1)(\mathcal{J}_{i,3}(s,\tau_2)-\mathcal{J}_{i,3}(s,\tau_1))\right|\geq\varepsilon/3\right)\to 0$$

This concludes the proof of the conditional stochastic equicontinuity.

Step 2. We focus on the one-dimension case and aim to show that, conditionally on data, for fixed $\tau\in\Upsilon$,

$$\frac{1}{\sqrt{n}}\sum_{s\in\mathcal{S}}\sum_{i=1}^n(\xi_i-1)\mathcal{J}_i(s,\tau)\rightsquigarrow\mathcal{N}(0,\tilde{\Sigma}_{sfe}(\tau,\tau)).$$

The finite-dimensional convergence can be established similarly by the Cramér-Wold device. In view of Lindeberg-Feller central limit theorem, we only need to show that (1)

$$\frac{1}{n}\sum_{i=1}^n[\sum_{s\in\mathcal{S}}\mathcal{J}_i(s,\tau)]^2\xrightarrow{p}\zeta_Y^2(\pi,\tau)+\tilde{\xi}_A'^2(\pi,\tau)+\xi_S^2(\pi,\tau)$$

and (2)

$$\frac{1}{n}\sum_{i=1}^n[\sum_{s\in\mathcal{S}}\mathcal{J}_i(s,\tau)]^2\mathbb{E}_\xi(\xi-1)^2 1\{|\sum_{s\in\mathcal{S}}(\xi_i-1)\mathcal{J}_i(s,\tau)|\geq\varepsilon\sqrt{n}\}\to 0.$$

(2) is obvious as $|\mathcal{J}_i(s,\tau)|$ is bounded and $\max_i|\xi_i-1|\lesssim\log(n)$ as ξ_i is sub-exponential. Next, we

focus on (1). We have

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}[\sum_{s\in\mathcal{S}}\mathcal{J}_i(s,\tau)]^2\\
=&\frac{1}{n}\sum_{i=1}^{n}\sum_{s\in\mathcal{S}}\Bigg\{\Bigg[\frac{A_i1\{S_i=s\}\eta_{i,1}(s,\tau)}{\pi f_1(q_1(\tau))}-\frac{(1-A_i)1\{S_i=s\}\eta_{i,0}(s,\tau)}{(1-\pi)f_0(q_0(\tau))}\Bigg]\\
&+F_1(s,\tau)(A_i-\pi)1\{S_i=s\}+\Bigg[\left(\frac{m_1(s,\tau)}{f_1(q_1(\tau))}-\frac{m_0(s,\tau)}{f_0(q_0(\tau))}\right)1\{S_i=s\}\Bigg]\Bigg\}^2\\
\equiv&\sigma_1^2+\sigma_2^2+\sigma_3^2+2\sigma_{12}+2\sigma_{13}+2\sigma_{23},
\end{aligned}
$$

where

$$
\sigma_1^2=\frac{1}{n}\sum_{s\in\mathcal{S}}\sum_{i=1}^{n}\Bigg[\frac{A_i1\{S_i=s\}\eta_{i,1}(s,\tau)}{\pi f_1(q_1(\tau))}-\frac{(1-A_i)1\{S_i=s\}\eta_{i,0}(s,\tau)}{(1-\pi)f_0(q_0(\tau))}\Bigg]^2,
$$

$$
\sigma_2^2=\frac{1}{n}\sum_{s\in\mathcal{S}}F_1^2(s,\tau)\sum_{i=1}^{n}(A_i-\pi)^2 1\{S_i=s\},
$$

$$
\sigma_3^2=\frac{1}{n}\sum_{i=1}^{n}\Bigg[\left(\frac{m_1(S_i,\tau)}{f_1(q_1(\tau))}-\frac{m_0(S_i,\tau)}{f_0(q_0(\tau))}\right)\Bigg]^2,
$$

$$
\sigma_{12}=\frac{1}{n}\sum_{i=1}^{n}\sum_{s\in\mathcal{S}}\Bigg[\frac{A_i1\{S_i=s\}\eta_{i,1}(s,\tau)}{\pi f_1(q_1(\tau))}-\frac{(1-A_i)1\{S_i=s\}\eta_{i,0}(s,\tau)}{(1-\pi)f_0(q_0(\tau))}\Bigg]F_1(s,\tau)(A_i-\pi)1\{S_i=s\},
$$

$$
\sigma_{13}=\frac{1}{n}\sum_{i=1}^{n}\sum_{s\in\mathcal{S}}\Bigg[\frac{A_i1\{S_i=s\}\eta_{i,1}(s,\tau)}{\pi f_1(q_1(\tau))}-\frac{(1-A_i)1\{S_i=s\}\eta_{i,0}(s,\tau)}{(1-\pi)f_0(q_0(\tau))}\Bigg]\Bigg[\left(\frac{m_1(s,\tau)}{f_1(q_1(\tau))}-\frac{m_0(s,\tau)}{f_0(q_0(\tau))}\right)\Bigg],
$$

and

$$
\sigma_{23}=\sigma_{12}=\frac{1}{n}\sum_{i=1}^{n}\sum_{s\in\mathcal{S}}F_1(s,\tau)(A_i-\pi)1\{S_i=s\}\Bigg[\left(\frac{m_1(s,\tau)}{f_1(q_1(\tau))}-\frac{m_0(s,\tau)}{f_0(q_0(\tau))}\right)\Bigg].
$$

For σ_1^2, we have

$$
\sigma_1^2=\frac{1}{n}\sum_{s\in\mathcal{S}}\sum_{i=1}^{n}\Bigg[\frac{A_i1\{S_i=s\}\eta_{i,1}^2(s,\tau)}{\pi^2 f_1^2(q_1(\tau))}-\frac{(1-A_i)1\{S_i=s\}\eta_{i,0}^2(s,\tau)}{(1-\pi)^2 f_0^2(q_0(\tau))}\Bigg]
$$

$$\stackrel{d}{=}\frac{1}{n}\sum_{s\in\mathcal{S}}\sum_{i=N(s)+1}^{N(s)+n_1(s)}\frac{\tilde{\eta}_{i,1}^2(s,\tau)}{\pi^2 f_1^2(q_1(\tau))}+\frac{1}{n}\sum_{s\in\mathcal{S}}\sum_{i=N(s)+n_1(s)+1}^{N(s)+n(s)}\frac{\tilde{\eta}_{i,0}^2(s,\tau)}{(1-\pi)^2 f_0^2(q_0(\tau))}$$
$$\xrightarrow{p}\frac{\tau(1-\tau)-\mathbb{E}m_1^s(S,\tau)}{\pi f_1^2(q_1(\tau))}+\frac{\tau(1-\tau)-\mathbb{E}m_0^s(S,\tau)}{(1-\pi)f_0^2(q_0(\tau))}=\zeta_Y^2(\pi,\tau),$$

where the second equality holds due to the rearrangement argument in Lemma E.2 and the convergence in probability holds due to uniform convergence of the partial sum process.

For σ_2^2, by Assumption 1,

$$\sigma_2^2=\frac{1}{n}\sum_{s\in\mathcal{S}}F_1^2(s,\tau)(D_n(s)-2\pi D_n(s)+\pi(1-\pi)1\{S_i=s\})\xrightarrow{p}\pi(1-\pi)\mathbb{E}F_1^2(S_i,\tau)=\tilde{\xi}_A'^2(\pi,\tau).$$

For σ_3^2, by the law of large number,

$$\sigma_3^2\xrightarrow{p}\mathbb{E}\left[\left(\frac{m_1(S_i,\tau)}{f_1(q_1(\tau))}-\frac{m_0(S_i,\tau)}{f_0(q_0(\tau))}\right)\right]^2=\xi_S^2(\pi,\tau).$$

For σ_{12}, we have

$$\sigma_{12}=\frac{1}{n}\sum_{s\in\mathcal{S}}(1-\pi)F_1(s,\tau)\sum_{i=1}^{n}\frac{A_i 1\{S_i=s\}\eta_{i,1}(s,\tau)}{\pi f_1(q_1(\tau))}-\frac{1}{n}\sum_{s\in\mathcal{S}}\pi F_1(s,\tau)\sum_{i=1}^{n}\frac{(1-A_i)1\{S_i=s\}\eta_{i,0}(s,\tau)}{(1-\pi)f_0(q_0(\tau))}$$
$$\stackrel{d}{=}\frac{1}{n}\sum_{s\in\mathcal{S}}(1-\pi)F_1(s,\tau)\sum_{i=N(s)+1}^{N(s)+n_1(s)}\frac{\tilde{\eta}_{i,1}(s,\tau)}{\pi f_1(q_1(\tau))}-\frac{1}{n}\sum_{s\in\mathcal{S}}\pi F_1(s,\tau)\sum_{i=N(s)+n_1(s)+1}^{N(s)+n(s)}\frac{\tilde{\eta}_{i,0}(s,\tau)}{(1-\pi)f_0(q_0(\tau))}\xrightarrow{p}0,$$

where the last convergence holds because by Lemma E.2,

$$\frac{1}{n}\sum_{i=N(s)+1}^{N(s)+n_1(s)}\tilde{\eta}_{i,1}(s,\tau)\xrightarrow{p}0\quad\text{and}\quad\frac{1}{n}\sum_{i=N(s)+n_1(s)+1}^{N(s)+n(s)}\tilde{\eta}_{i,0}(s,\tau)\xrightarrow{p}0.$$

By the same argument, we can show that

$$\sigma_{13}\xrightarrow{p}0.$$

Last, for σ_{23}, by Assumption 1,

$$\sigma_{23}=\sum_{s\in\mathcal{S}}F_1(s,\tau)\left[\left(\frac{m_1(s,\tau)}{f_1(q_1(\tau))}-\frac{m_0(s,\tau)}{f_0(q_0(\tau))}\right)\right]\frac{D_n(s)}{n}\xrightarrow{p}0.$$

Therefore, we have

$$\frac{1}{n}\sum_{i=1}^{n}[\sum_{s\in\mathcal{S}}\mathcal{J}_i(s,\tau)]^2\xrightarrow{p}\zeta_Y^2(\pi,\tau)+\tilde{\xi}_A'^2(\pi,\tau)+\xi_S^2(\pi,\tau).$$

$\square$

Lemma S.E.6. *Recall $R^*_{sfe,2,1}(u,\tau)$ and $R^*_{sfe,2,0}(u,\tau)$ defined in (S.D.5) and (S.D.6), respectively. If Assumptions in Theorem 5.1 hold, then (S.D.5) and (S.D.6) hold and*

$$\sup_{\tau\in\Upsilon}|R^*_{sfe,2,1}(u,\tau)| = o_p(1) \quad \textit{and} \quad \sup_{\tau\in\Upsilon}|R^*_{sfe,2,0}(u,\tau)| = o_p(1).$$

Proof. We focus on (S.D.5). Following the definition of M_{ni} in the proof of Lemma E.5 and the argument in the Step 1.2 of the proof of Theorem S.A.1, we have

$$\begin{aligned}
&L^*_{2,1,n}(u,\tau)\\
=&\sum_{s\in\mathcal{S}}\sum_{i=N(s)+1}^{N(s)+n_1(s)} M_{ni}\int_0^{\frac{u'\iota_1}{\sqrt{n}}-\frac{E^*_n(s)}{\sqrt{n}}\left(q(\tau)+\frac{u_1}{\sqrt{n}}\right)}\left(1\{Y_i^s(1)\le q_1(\tau)+v\}-1\{Y_i^s(1)\le q_1(\tau)\}\right)dv\\
=&\sum_{s\in\mathcal{S}}\sum_{i=N(s)+1}^{N(s)+n_1(s)} M_{ni}\left[\phi_i(u,\tau,s,E^*_n(s))-\mathbb{E}\phi_i(u,\tau,E^*_n(s))\right]+\sum_{s\in\mathcal{S}}\sum_{i=N(s)+1}^{N(s)+n_1(s)} M_{ni}\mathbb{E}\phi_i(u,\tau,s,E^*_n(s)),
\end{aligned} \tag{S.E.12}$$

where $E^*_n(s)=\sqrt{n}(\hat{\pi}^*(s)-\pi)=\frac{n}{n^*(s)}\frac{D^*_n(s)}{\sqrt{n}}=O_p(1)$,

$$\phi_i(u,\tau,s,e)=\int_0^{\frac{u'\iota_1}{\sqrt{n}}-\frac{e}{\sqrt{n}}\left(q(\tau)+\frac{u_1}{\sqrt{n}}\right)}\left(1\{Y_i^s(1)\le q_1(\tau)+v\}-1\{Y_i^s(1)\le q_1(\tau)\}\right)dv,$$

and $\mathbb{E}\phi_i(u,\tau,s,E^*_n(s))$ is interpreted as $\mathbb{E}\phi_i(u,\tau,s,e)$ with e being evaluated at $E^*_n(s)$.

For the first term on the RHS of (S.E.12), similar to (E.11), we have

$$\begin{aligned}
&\sum_{s\in\mathcal{S}}\sum_{i=N(s)+1}^{N(s)+n_1(s)} M_{ni}\left[\phi_i(u,\tau,s,E^*_n(s))-\mathbb{E}\phi_i(u,\tau,s,E^*_n(s))\right]\\
=&\sum_{s\in\mathcal{S}}\sum_{i=N(s)+1}^{N(s)+n_1(s)} \xi_i^s\left[\phi_i(u,\tau,s,E^*_n(s))-\mathbb{E}\phi_i(u,\tau,s,E^*_n(s))\right]+\sum_{s\in\mathcal{S}} r_n(u,\tau,s,E^*_n(s)),
\end{aligned} \tag{S.E.13}$$

where $\{\xi_i^s\}_{i=1}^n$ is a sequence of i.i.d. Poisson(1) random variables and is independent of everything else, and

$$r_n(u,\tau,s,e)=\text{sign}(N(n_1(s))-n_1(s))\sum_{j=1}^{\infty}\frac{\#I_n^j(s)}{\sqrt{n}}\frac{1}{\#I_n^j(s)}\sum_{i\in I_n^j(s)}\sqrt{n}\left[\phi_i(u,\tau,s,e)-\mathbb{E}\phi_i(u,\tau,s,e)\right].$$

We aim to show

$$\sup_{|e|\leq M,\tau\in\Upsilon,s\in\mathcal{S}} |r_n(u,\tau,s,e)| = o_p(1), \tag{S.E.14}$$

Recall that the proof of Lemma E.5 relies on (E.10) and the fact that

$$\mathbb{E}\sup_{n\geq k\geq n_0}\sup_{\tau\in\Upsilon,s\in\mathcal{S}}\left|\frac{1}{k}\sum_{j=1}^{k}\tilde{\eta}_{j,1}(s,\tau)\right| \to 0.$$

Using the same argument and replacing $\tilde{\eta}_{j,1}(s,\tau)$ by $\sqrt{n}\,[\phi_i(u,\tau,s,e)-\mathbb{E}\phi_i(u,\tau,s,e)]$, in order to show (S.E.14), we only need to verify that, as $n\to\infty$ followed by $n_0\to\infty$,

$$\mathbb{E}\sup_{n\geq k\geq n_0}\sup_{|e|\leq M,\tau\in\Upsilon,s\in\mathcal{S}}\left|\frac{1}{k}\sum_{j=1}^{k}\sqrt{n}\,[\phi_i(u,\tau,s,e)-\mathbb{E}\phi_i(u,\tau,s,e)]\right| \to 0$$

Because $\sup_{|e|\leq M,\tau\in\Upsilon,s\in\mathcal{S}}\left|\frac{1}{k}\sum_{j=1}^{k}\sqrt{n}\,[\phi_i(u,\tau,s,e)-\mathbb{E}\phi_i(u,\tau,s,e)]\right|$ is bounded as shown below, it suffices to show that, for any $\varepsilon>0$, as $n\to\infty$ followed by $n_0\to\infty$,

$$\mathbb{P}\left(\sup_{n\geq k\geq n_0}\sup_{|e|\leq M,\tau\in\Upsilon,s\in\mathcal{S}}\left|\frac{1}{k}\sum_{j=1}^{k}\sqrt{n}\,[\phi_i(u,\tau,s,e)-\mathbb{E}\phi_i(u,\tau,s,e)]\right|\geq\varepsilon\right)\to 0. \tag{S.E.15}$$

Define the class of functions $\mathcal{F}_n$ as

$$\mathcal{F}_n = \{\sqrt{n}\,[\phi_i(u,\tau,s,e)-\mathbb{E}\phi_i(u,\tau,s,e)] : |e|\leq M, \tau\in\Upsilon, s\in\mathcal{S}\}.$$

Then, $\mathcal{F}_n$ is nested by a VC-class with fixed VC-index. In addition, for fixed u, $\mathcal{F}_n$ has a bounded (and independent of n) envelope function

$$F = |u'\iota_1| + M\left(\max_{\tau\in\Upsilon}|q(\tau)| + |u_1|\right).$$

Last, define $\mathcal{I}_l = \{2^l, 2^l+1,\cdots,2^{l+1}-1\}$. Then,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{n\geq k\geq n_0}\sup_{|e|\leq M,\tau\in\Upsilon,s\in\mathcal{S}}\left|\frac{1}{k}\sum_{j=1}^{k}\sqrt{n}\,[\phi_i(u,\tau,s,e)-\mathbb{E}\phi_i(u,\tau,s,e)]\right|\geq\varepsilon\right)\\
&\leq\sum_{l=\lfloor\log_2(n_0)\rfloor}^{\lfloor\log_2(n)\rfloor+1}\mathbb{P}\left(\sup_{k\in\mathcal{I}_l}\sup_{|e|\leq M,\tau\in\Upsilon,s\in\mathcal{S}}\left|\frac{1}{k}\sum_{j=1}^{k}\sqrt{n}\,[\phi_i(u,\tau,s,e)-\mathbb{E}\phi_i(u,\tau,s,e)]\right|\geq\varepsilon\right)\\
&\leq\sum_{l=\lfloor\log_2(n_0)\rfloor}^{\lfloor\log_2(n)\rfloor+1}\mathbb{P}\left(\sup_{k\leq 2^{l+1}}\sup_{|e|\leq M,\tau\in\Upsilon,s\in\mathcal{S}}\left|\sum_{j=1}^{k}\sqrt{n}\,[\phi_i(u,\tau,s,e)-\mathbb{E}\phi_i(u,\tau,s,e)]\right|\geq\varepsilon 2^l\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{l=\lfloor \log_2(n_0)\rfloor}^{\lfloor \log_2(n)\rfloor+1} 9\mathbb{P}\left(\sup_{|e|\leq M,\tau\in\Upsilon,s\in\mathcal{S}}\left|\sum_{j=1}^{2^{l+1}}\sqrt{n}\left[\phi_i(u,\tau,s,e)-\mathbb{E}\phi_i(u,\tau,s,e)\right]\right|\geq \varepsilon 2^l/30\right)\\
&\leq \sum_{l=\lfloor \log_2(n_0)\rfloor}^{\lfloor \log_2(n)\rfloor+1} \frac{270\mathbb{E}\sup_{|e|\leq M,\tau\in\Upsilon,s\in\mathcal{S}}\left|\sum_{j=1}^{2^{l+1}}\sqrt{n}\left[\phi_i(u,\tau,s,e)-\mathbb{E}\phi_i(u,\tau,s,e)\right]\right|}{\varepsilon 2^l}\\
&\leq \sum_{l=\lfloor \log_2(n_0)\rfloor}^{\lfloor \log_2(n)\rfloor+1} \frac{C_1}{\varepsilon 2^{l/2}}\\
&\leq \frac{2C_1}{\varepsilon\sqrt{n_0}} \to 0,
\end{aligned}
$$

where the first inequality holds by the union bound, the second inequality holds because on $\mathcal{I}_l$, $2^{l+1}\geq k\geq 2^l$, the third inequality follows the same argument in the proof of Theorem 3.1, the fourth inequality is due to the Markov inequality, the fifth inequality follows the standard maximal inequality such as van der Vaart and Wellner (1996, Theorem 2.14.1) and the constant C_1 is independent of (l,ε,n), and the last inequality holds by letting $n\to\infty$. Because ε is arbitrary, we have established (S.E.15), and thus, (S.E.14), which further implies that

$$
\sup_{\tau\in\Upsilon,s\in\mathcal{S}}|r_n(u,\tau,s,E_n^*(s))| = o_p(1),
$$

For the leading term of (S.E.13), we have

$$
\begin{aligned}
&\sum_{s\in\mathcal{S}}\sum_{i=N(s)+1}^{N(s)+n_1(s)}\xi_i^s\left[\phi_i(u,\tau,s,E_n^*(s))-\mathbb{E}\phi_i(u,\tau,s,E_n^*(s))\right]\\
=&\sum_{s\in\mathcal{S}}\left[\Gamma_n^{s*}(N(s),\tau,E_n^*(s))-\Gamma_n^{s*}(N(s)+n_1(s),\tau,E_n^*(s))\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma_n^{s*}(k,\tau,e) = &\sum_{i=1}^{k}\xi_i^s\int_0^{\frac{u'\iota_1-e(q(\tau)+\frac{u_1}{\sqrt{n}})}{\sqrt{n}}}\left(1\{Y_i^s(1)\leq q_1(\tau)+v\}-1\{Y_i^s(1)\leq q_1(\tau)\}\right)dv\\
&-k\mathbb{E}\left[\int_0^{\frac{u'\iota_1-e(q(\tau)+\frac{u_1}{\sqrt{n}})}{\sqrt{n}}}\left(1\{Y_i^s(1)\leq q_1(\tau)+v\}-1\{Y_i^s(1)\leq q_1(\tau)\}\right)dv\right].
\end{aligned}
$$

By the same argument in (S.B.8), we can show that

$$
\sup_{0<t\leq 1,\tau\in\Upsilon,|e|\leq M}|\Gamma_n^{s*}(k,\tau,e)| = o_p(1),
$$

where we need to use the fact that the Poisson(1) random variable has an exponential tail and thus

$$\mathbb{E}\sup_{i\in\{1,\cdots,n\},s\in\mathcal{S}}\xi_i^s = O(\log(n)).$$

Therefore,

$$\sup_{\tau\in\Upsilon}\left|\sum_{s\in\mathcal{S}}\sum_{i=N(s)+1}^{N(s)+n_1(s)}M_{ni}\left[\phi_i(u,\tau,s,E_n^*(s))-\mathbb{E}\phi_i(u,\tau,E_n^*(s))\right]\right| = o_p(1). \tag{S.E.16}$$

For the second term on the RHS of (S.E.12), we have

$$\begin{aligned}\sum_{s\in\mathcal{S}}\sum_{i=N(s)+1}^{N(s)+n_1(s)}M_{ni}\mathbb{E}\phi_i(u,\tau,s,e) &= \sum_{s\in\mathcal{S}}n_1^*(s)\mathbb{E}\phi_i(u,\tau,s,e)\\ &= \sum_{s\in\mathcal{S}}\pi p(s)\frac{f_1(q_1(\tau)|s)}{2}(u'\iota_1 - eq(\tau))^2 + o(1),\end{aligned} \tag{S.E.17}$$

where the $o(1)$ term holds uniformly over $(\tau, e) \in \Upsilon \times [-M, M]$, the first equality holds because $\sum_{i=N(s)+1}^{N(s)+n_1(s)} M_{ni} = n_1^*(s)$ and the second equality holds by the same calculation in (S.B.10) and the facts that $n^*(s)/n \xrightarrow{p} p(s)$ and

$$\frac{n_1^*(s)}{n} = \frac{D_n^*(s)+\pi n^*(s)}{n} \xrightarrow{p} \pi p(s).$$

Combining (S.D.5), (S.E.12), (S.E.16), (S.E.17), and the facts that $E_n^*(s) = \frac{n}{n^*(s)}\frac{D_n^*(s)}{\sqrt{n}}$ and $\frac{n}{n^*(s)} \xrightarrow{p} 1/p(s)$, we have

$$L_{2,1,n}^*(u,\tau) = \frac{\pi f_1(q_1(\tau))}{2}(u'\iota_1)^2 - \sum_{s\in\mathcal{S}}f_1(q_1(\tau)|s)\frac{\pi D_n^*(s)u'\iota_1}{\sqrt{n}}q(\tau) + h_{2,1}^*(\tau) + R_{sfe,2,1}^*(u,\tau),$$

where

$$h_{2,1}^*(\tau) = \sum_{s\in\mathcal{S}}\frac{\pi f_1(q_1(\tau)|s)}{2}p(s)(E_n^*(s))^2q^2(\tau)$$

and

$$\sup_{\tau\in\Upsilon}|R_{sfe,2,1}^*(u,\tau)| = o_p(1).$$

This concludes the proof. ☐

Lemma S.E.7. *Recall the definition of* $(W_{sfe,n,1}^*(\tau) - \mathcal{W}_{n,1}(\tau), W_{sfe,n,2}^*(\tau), W_{sfe,n,3}^*(\tau) - \mathcal{W}_{n,2}(\tau))$

in (S.D.7). *If Assumptions in Theorem 5.1 hold, then conditionally on data,*

$$(W^*_{sfe,n,1}(\tau) - \mathcal{W}_{n,1}(\tau), W^*_{sfe,n,2}(\tau), W^*_{sfe,n,3}(\tau) - \mathcal{W}_{n,2}(\tau)) \rightsquigarrow (\mathcal{B}_1(\tau), \mathcal{B}_2(\tau), \mathcal{B}_3(\tau)),$$

where $(\mathcal{B}_1(\tau), \mathcal{B}_2(\tau), \mathcal{B}_3(\tau))$ *are three independent Gaussian processes with covariance kernels*

$$\Sigma_1(\tau_1, \tau_2) = \frac{\min(\tau_1,\tau_2) - \tau_1\tau_2 - \mathbb{E}m_1(S,\tau_1)m_1(S,\tau_2)}{\pi f_1(q_1(\tau_1))f_1(q_1(\tau_2))} + \frac{\min(\tau_1,\tau_2) - \tau_1\tau_2 - \mathbb{E}m_0(S,\tau_1)m_0(S,\tau_2)}{(1-\pi)f_0(q_0(\tau_1))f_0(q_0(\tau_2))},$$

$$\begin{aligned}
&\Sigma_2(\tau_1, \tau_2) \\
=&\mathbb{E}\gamma(S)\left[(m_1(S,\tau_1) - m_0(S,\tau_1))\left(\frac{1-\pi}{\pi f_1(q_1(\tau_1))} - \frac{\pi}{(1-\pi)f_0(q_0(\tau_1))}\right) + q(\tau_1)\left(\frac{f_1(q(\tau_1)|S)}{f_1(q_1(\tau_1))} - \frac{f_0(q(\tau_1)|S)}{f_0(q_0(\tau_1))}\right)\right] \\
&\times\left[(m_1(S,\tau_2) - m_0(S,\tau_2))\left(\frac{1-\pi}{\pi f_1(q_1(\tau_2))} - \frac{\pi}{(1-\pi)f_0(q_0(\tau_2))}\right) + q(\tau_2)\left(\frac{f_1(q(\tau_2)|S)}{f_1(q_2(\tau_2))} - \frac{f_0(q(\tau_2)|S)}{f_0(q_0(\tau_2))}\right)\right],
\end{aligned}$$

and

$$\Sigma_3(\tau_1, \tau_2) = \mathbb{E}\left[\frac{m_1(S,\tau_1)}{f_1(q_1(\tau_1))} - \frac{m_0(S,\tau_1)}{f_0(q_0(\tau_1))}\right]\left[\frac{m_1(S,\tau_2)}{f_1(q_1(\tau_2))} - \frac{m_0(S,\tau_2)}{f_0(q_0(\tau_2))}\right],$$

respectively.

Proof. Let $\mathcal{A}_n = \{(A_i^*, S_i^*, A_i, S_i) : i = 1, \cdots, n\}$. Following the definition of M_{ni} and arguments in the proof of Lemma E.5, we have

$$\begin{aligned}
&\{W^*_{sfe,n,1}(\tau) - \mathcal{W}_{n,1}(\tau)|\mathcal{A}_n\} \\
\stackrel{d}{=}&\left\{\sum_{s\in\mathcal{S}}\frac{1}{\sqrt{n}}\left[\sum_{i=N(s)+1}^{N(s)+n_1(s)}(M_{ni}-1)\left(\frac{\tilde{\eta}_{i,1}(s,\tau)}{\pi f_1(q_1(\tau))}\right) - \sum_{i=N(s)+n_1(s)+1}^{N(s)+n(s)}(M_{ni}-1)\left(\frac{\tilde{\eta}_{i,0}(s,\tau)}{(1-\pi)f_0(q_0(\tau))}\right)\right]\middle|\mathcal{A}_n\right\} \\
=&\left\{\sum_{s\in\mathcal{S}}\frac{1}{\sqrt{n}}\left[\sum_{i=N(s)+1}^{N(s)+n_1(s)}(\xi_i^s-1)\frac{\tilde{\eta}_{i,1}(s,\tau)}{\pi f_1(q_1(\tau))} - \sum_{i=N(s)+n_1(s)+1}^{N(s)+n(s)}(\xi_i^s-1)\frac{\tilde{\eta}_{i,0}(s,\tau)}{(1-\pi)f_0(q_0(\tau))}\right] + R_1(\tau)\middle|\mathcal{A}_n\right\},
\end{aligned}$$

where $\sup_{\tau\in\Upsilon}|R_1(\tau)| = o_p(1)$ and $\{\xi_i^s\}_{i=1}^n$, $s \in \mathcal{S}$ are sequences of i.i.d. Poisson(1) random variables that are independent of $\mathcal{A}_n$ and across $s \in \mathcal{S}$. In addition, by the same argument in the proof of Lemma E.2, we have

$$\begin{aligned}
&\sum_{s\in\mathcal{S}}\frac{1}{\sqrt{n}}\left[\sum_{i=N(s)+1}^{N(s)+n_1(s)}(\xi_i^s-1)\frac{\tilde{\eta}_{i,1}(s,\tau)}{\pi f_1(q_1(\tau))} - \sum_{i=N(s)+n_1(s)+1}^{N(s)+n(s)}(\xi_i^s-1)\frac{\tilde{\eta}_{i,0}(s,\tau)}{(1-\pi)f_0(q_0(\tau))}\right] \\
=&\sum_{s\in\mathcal{S}}\frac{1}{\sqrt{n}}\left[\sum_{i=\lfloor nF(s)\rfloor+1}^{\lfloor n(F(s)+\pi p(s))\rfloor}(\xi_i^s-1)\frac{\tilde{\eta}_{i,1}(s,\tau)}{\pi f_1(q_1(\tau))} - \sum_{i=\lfloor n(F(s)+\pi p(s))\rfloor+1}^{\lfloor n(F(s)+p(s))\rfloor}(\xi_i^s-1)\frac{\tilde{\eta}_{i,0}(s,\tau)}{(1-\pi)f_0(q_0(\tau))}\right] + R_2(\tau) \\
\equiv&W_1^*(\tau) + R_2(\tau),
\end{aligned}$$

where $\sup_{\tau\in\Upsilon}|R_2(\tau)| = o_p(1)$. Because both $W^*_{sfe,n,2}(\tau)$ and $W^*_{sfe,n,3}(\tau) - \mathcal{W}_{n,2}(\tau)$ are in the σ-field generated by $\mathcal{A}_n$, we have

$$(W^*_{sfe,n,1}(\tau) - \mathcal{W}_{n,1}(\tau), W^*_{sfe,n,2}(\tau), W^*_{sfe,n,3}(\tau) - \mathcal{W}_{n,2}(\tau))$$
$$\stackrel{d}{=}(W^*_1(\tau) + R_1(\tau) + R_2(\tau), W^*_{sfe,n,2}(\tau), W^*_{sfe,n,3}(\tau) - \mathcal{W}_{n,2}(\tau)).$$

In addition, note that $\{\xi^s_i\}_{i=1}^n$ and $\{\tilde{\eta}_{i,1}(s,\tau), \tilde{\eta}_{i,1}(s,\tau)\}_{i=1}^n$ are independent of $\mathcal{A}_n$, therefore, $W^*_1(\tau) \perp\!\!\!\perp (W^*_{sfe,n,2}(\tau), W^*_{sfe,n,3}(\tau) - \mathcal{W}_{n,2}(\tau))$. Applying van der Vaart and Wellner (1996, Theorem 2.9.6) to each segment

$$\lfloor nF(s)\rfloor + 1, \cdots, \lfloor n(F(s) + \pi p(s))\rfloor \quad \text{or} \quad \lfloor n(F(s) + \pi p(s))\rfloor + 1, \cdots, \lfloor n(F(s) + p(s))\rfloor$$

for $s \in \mathcal{S}$ and noticing that $\{\tilde{\eta}_{i,1}(s,\tau)\}_{i=1}^n$ and $\{\tilde{\eta}_{i,0}(s,\tau)\}_{i=1}^n$ are two i.i.d. sequences for each $s \in \mathcal{S}$, independent of each other, and independent across s, we have, conditionally on $\{\tilde{\eta}_{i,1}(s,\tau), \tilde{\eta}_{i,0}(s,\tau)\}_{i=1}^n$, $s \in \mathcal{S}$,

$$W^*_1(\tau) \rightsquigarrow \mathcal{B}_1(\tau)$$

with the covariance kernel $\Sigma_1(\tau_1, \tau_2)$.

For $W^*_{sfe,n,2}(\tau)$, we note that it depends on data only through $\{S^*_i\}_{i=1}^n$. By Assumption 4,

$$W^*_{sfe,n,2}(\tau)|\{S^*_i\}_{i=1}^n \rightsquigarrow \mathcal{B}_2(\tau)$$

with the covariance kernel $\Sigma_2(\tau_1, \tau_2)$.

Last, for $W^*_{sfe,n,3}(\tau) - \mathcal{W}_{n,2}(\tau)$, note that $\{S^*_i\}$ is sampled by the standard bootstrap procedure. Therefore, directly applying van der Vaart and Wellner (1996, Theorem 3.6.2), we have

$$W^*_{sfe,n,3}(\tau) - \mathcal{W}_{n,2}(\tau) = \frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi'_i - 1)\left[\frac{m_1(S_i,\tau)}{f_1(q_1(\tau))} - \frac{m_0(S_i,\tau)}{f_0(q_0(\tau))}\right] + R_3(\tau)$$

where $\sup_{\tau\in\Upsilon}|R_3(\tau)| = o_p(1)$, $\{\xi'_i\}_{i=1}^n$ is a sequence of i.i.d. Poisson(1) random variables that is independent of data and $\{\xi^s_i\}_{i=1}^n$, $s \in \mathcal{S}$. By van der Vaart and Wellner (1996, Theorem 3.6.2), conditionally on data $\{S_i\}_{i=1}^n$,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi'_i - 1)\left[\frac{m_1(S_i,\tau)}{f_1(q_1(\tau))} - \frac{m_0(S_i,\tau)}{f_0(q_0(\tau))}\right] \rightsquigarrow \mathcal{B}_3(\tau),$$

where $\mathcal{B}_3(\tau)$ has the covariance kernel $\Sigma_3(\tau_1, \tau_2)$. Furthermore, $\mathcal{B}_2(\tau)$ and $\mathcal{B}_3(\tau)$ are independent as $\Sigma_2(\tau_1, \tau_2)$ is not a function of $\{S^*_i\}_{i=1}^n$. This concludes the proof.

□

S.F Additional Simulation Results

S.F.1 DGPs

We consider the following four DGPs with parameters $\gamma = 4$, $\sigma = 2$, and μ which will be specified later. DGPs 1 and 3 correspond to DGPs 1 and 2 in Section 6 in the main paper.

1. Let Z be the standardized Beta(2, 2) distributed, $S_i = \sum_{j=1}^{4}\{Z_i \leq g_j\}$, and $(g_1, \cdots, g_4) = (-0.25\sqrt{20}, 0, 0.25\sqrt{20}, 0.5\sqrt{20})$. The outcome equation is

$$Y_i = A_i\mu + \gamma Z_i + \eta_i,$$

where $\eta_i = \sigma A_i\varepsilon_{i,1} + (1 - A_i)\varepsilon_{i,2}$ and $(\varepsilon_{i,1}, \varepsilon_{i,2})$ are jointly standard normal.

2. Let S be the same as in DGP1. The outcome equation is

$$Y_i = A_i\mu + \gamma Z_i A_i - \gamma(1 - A_i)(\log(Z_i + 3)1\{Z_i \leq 0.5\}) + \eta_i.$$

where $\eta_i = \sigma A_i\varepsilon_{i,1} + (1 - A_i)\varepsilon_{i,2}$ and $(\varepsilon_{i,1}, \varepsilon_{i,2})$ are jointly standard normal.

3. Let Z be uniformly distributed on $[-2, 2]$, $S_i = \sum_{j=1}^{4}\{Z_i \leq g_j\}$, and $(g_1, \cdots, g_4) = (-1, 0, 1, 2)$. The outcome equation is

$$Y_i = A_i\mu + A_i m_{i,1} + (1 - A_i)m_{i,0} + \eta_i,$$

where $m_{i,0} = \gamma Z_i^2 1\{|Z_i| \geq 1\} + \frac{\gamma}{4}(2 - Z_i^2)1\{|Z_i| < 1\}$, $\eta_i = \sigma(1 + Z_i^2)A_i\varepsilon_{i,1} + (1 + Z_i^2)(1 - A_i)\varepsilon_{i,2}$, and $(\varepsilon_{i,1}, \varepsilon_{i,2})$ are mutually independent $T(3)/3$ distributed.

4. Let Z_i be normally distributed with mean 0 and variance 4, $S_i = \sum_{j=1}^{4}\{Z_i \leq g_j\}$, $(g_1, \cdots, g_4) = (2\Phi^{-1}(0.25), 2\Phi^{-1}(0.5), 2\Phi^{-1}(0.75), \infty)$, and $\Phi(\cdot)$ is the standard normal CDF. The outcome equation is

$$Y_i = A_i\mu + A_i m_{i,1} + (1 - A_i)m_{i,0} + \eta_i,$$

where $m_{i,0} = -\gamma Z_i^2/4$, $m_{i,1} = \gamma Z_i^2/4$,

$$\eta_i = \sigma(1 + 0.5\exp(-Z_i^2/2))A_i\varepsilon_{i,1} + (1 + 0.5\exp(-Z_i^2/2))(1 - A_i)\varepsilon_{i,2},$$

and $(\varepsilon_{i,1}, \varepsilon_{i,2})$ are jointly standard normal.

When $\pi = \frac{1}{2}$, for each DGP, we consider four randomization schemes:

1. SRS: Treatment assignment is generated as in Example 1.

2. WEI: Treatment assignment is generated as in Example 2 with $\phi(x) = (1 - x)/2$.

3. BCD: Treatment assignment is generated as in Example 3 with $\lambda = 0.75$.

4. SBR: Treatment assignment is generated as in Example 4.

When $\pi \neq 0.5$, we focus on SRS and SBR. We conduct the simulations with sample sizes $n = 200$ and 400. The numbers of simulation replications and bootstrap samples are 1000. Under the null, $\mu = 0$ and the true parameters of interest are computed by simulations with 10^6 sample size and 10^4 replications. Under the alternative, we perturb the true values by $\mu = 1$ and $\mu = 0.75$ for $n = 200$ and 400, respectively. We consider the following eight t-statistics.

1. "s/naive": the point estimator is computed by the simple QR and its standard error σ_{naive} is computed as

$$
\begin{aligned}
\sigma^2_{naive} =& \frac{\tau(1-\tau) - \frac{1}{n}\sum_{i=1}^n \hat{m}_1^2(S_i,\tau)}{\pi \hat{f}_1^2(\hat{q}_1(\tau))} + \frac{\tau(1-\tau) - \frac{1}{n}\sum_{i=1}^n \hat{m}_0^2(S_i,\tau)}{(1-\pi)\hat{f}_0^2(\hat{q}_0(\tau))} \\
&+ \frac{1}{n}\sum_{i=1}^n \pi(1-\pi)\left(\frac{\hat{m}_1(S_i,\tau)}{\pi \hat{f}_1(\hat{q}_1(\tau))} + \frac{\hat{m}_0(S_i,\tau)}{(1-\pi)\hat{f}_0(\hat{q}_0(\tau))}\right)^2 \\
&+ \frac{1}{n}\sum_{i=1}^n \left(\frac{\hat{m}_1(S_i,\tau)}{\hat{f}_1(\hat{q}_1(\tau))} - \frac{\hat{m}_0(S_i,\tau)}{\hat{f}_0(\hat{q}_0(\tau))}\right)^2, \qquad \text{(S.F.1)}
\end{aligned}
$$

where $\hat{q}_j(\tau)$ is the τ-the empirical quantile of $Y_i|A_i = j$,

$$\hat{m}_{i,1}(s,\tau) = \frac{\sum_{i=1}^n A_i 1\{S_i = s\}(\tau - 1\{Y_i \leq \hat{q}_1(\tau)\})}{n_1(s)},$$

$$\hat{m}_{i,0}(s,\tau) = \frac{\sum_{i=1}^n (1-A_i) 1\{S_i = s\}(\tau - 1\{Y_i \leq \hat{q}_0(\tau)\})}{n(s) - n_1(s)},$$

and for $j = 0, 1$, $\hat{f}_j(\cdot)$ is computed by the kernel density estimation using the observations Y_i provided that $A_i = j$, bandwidth $h_j = 1.06\hat{\sigma}_j n_j^{-1/5}$, and the Gaussian kernel function, where $\hat{\sigma}_j$ is the standard deviation of the observations Y_i provided that $A_i = j$, and $n_j = \sum_{i=1}^n \{A_i = j\}$, $j = 0, 1$.

2. "s/adj": exactly the same as the "s/naive" method with one difference: replacing $\pi(1-\pi)$ in σ^2_{naive} by $\gamma(S_i)$.

3. "s/W": the point estimator is computed by the simple QR and its standard error σ_B is computed by the weighted bootstrap procedure. The bootstrap weights $\{\xi_i\}_{i=1}^n$ are generated from the standard exponential distribution. Denote $\{\hat{\beta}^w_{1,b}\}_{b=1}^B$ as the collection of B estimates obtained by the simple QR applied to the samples generated by the weighted bootstrap procedure. Then,

$$\sigma_B = \frac{\hat{Q}(0.9) - \hat{Q}(0.1)}{\Phi^{-1}(0.9) - \Phi^{-1}(0.1)},$$

where $\Phi(\cdot)$ is the standard normal CDF and $\hat{Q}(\tau)$ is the τ-th empirical quantile of $\{\hat{\beta}^w_{1,b}\}_{b=1}^B$.

4. "sfe/W": the same as above with one difference: the estimation method for both the original and bootstrap samples is the QR with strata fixed effects.

5. "ipw/W": the same as above with one difference: the estimation method for both the original and bootstrap samples is the inverse propensity score weighted QR.

6. "s/CA": the point estimator is computed by the simple QR and its standard error σ_{CA} is computed by the covariate-adaptive bootstrap procedure. Denote $\{\hat{\beta}^*_{1,b}\}_{b=1}^B$ as the collection of B estimates obtained by the simple QR applied to the samples generated by the covariate-adaptive bootstrap procedure. Then,

$$\sigma_{CA} = \frac{\hat{Q}(0.9) - \hat{Q}(0.1)}{\Phi^{-1}(0.9) - \Phi^{-1}(0.1)},$$

where $\hat{Q}(\tau)$ is the τ-th empirical quantile of $\{\hat{\beta}^*_{1,b}\}_{b=1}^B$.

7. "sfe/CA": the same as above with one difference: the estimation method for both the original and bootstrap samples is the QR with strata fixed effects.

8. "ipw/CA": the same as above with one difference: the estimation method for both the original and bootstrap samples is the inverse propensity score weighted QR.

S.F.2 QTE, H_0, $\pi = 0.5$

Table 1: H_0, $n = 200$, $\tau = 0.25$

M	A	s/naive	s/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.042	0.042	0.051	0.039	0.047	0.046	0.044	0.046
	WEI	0.011	0.038	0.018	0.043	0.046	0.037	0.047	0.047
	BCD	0.004	0.041	0.010	0.043	0.043	0.045	0.048	0.048
	SBR	0.003	0.047	0.003	0.047	0.054	0.049	0.046	0.046
2	SRS	0.045	0.045	0.060	0.062	0.066	0.056	0.069	0.069
	WEI	0.023	0.037	0.049	0.056	0.066	0.068	0.064	0.068
	BCD	0.021	0.037	0.032	0.049	0.057	0.063	0.059	0.057
	SBR	0.025	0.042	0.037	0.050	0.054	0.057	0.054	0.053
3	SRS	0.042	0.042	0.045	0.045	0.054	0.055	0.044	0.058
	WEI	0.042	0.043	0.037	0.044	0.045	0.045	0.043	0.045
	BCD	0.052	0.056	0.044	0.050	0.057	0.057	0.057	0.055
	SBR	0.046	0.053	0.041	0.043	0.048	0.052	0.048	0.047
4	SRS	0.054	0.054	0.048	0.046	0.049	0.046	0.043	0.048
	WEI	0.050	0.051	0.045	0.035	0.047	0.051	0.043	0.055
	BCD	0.056	0.059	0.040	0.030	0.049	0.047	0.044	0.048
	SBR	0.061	0.065	0.044	0.032	0.053	0.057	0.051	0.053

Table 2: H_0, $n = 200$, $\tau = 0.5$

M	A	s/naive	s/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.045	0.045	0.047	0.043	0.044	0.044	0.039	0.039
	WEI	0.012	0.040	0.014	0.044	0.043	0.037	0.041	0.035
	BCD	0.002	0.057	0.003	0.040	0.041	0.044	0.039	0.039
	SBR	0.001	0.057	0.001	0.045	0.046	0.045	0.045	0.044
2	SRS	0.045	0.045	0.057	0.066	0.061	0.048	0.064	0.066
	WEI	0.033	0.065	0.037	0.056	0.065	0.065	0.056	0.061
	BCD	0.022	0.062	0.027	0.048	0.056	0.057	0.057	0.054
	SBR	0.017	0.050	0.017	0.040	0.046	0.048	0.048	0.046
3	SRS	0.004	0.004	0.047	0.045	0.052	0.052	0.047	0.053
	WEI	0.006	0.006	0.045	0.050	0.058	0.052	0.053	0.057
	BCD	0.010	0.010	0.045	0.050	0.051	0.050	0.050	0.053
	SBR	0.008	0.011	0.048	0.048	0.053	0.046	0.051	0.047
4	SRS	0.013	0.013	0.050	0.036	0.051	0.055	0.035	0.043
	WEI	0.011	0.011	0.043	0.033	0.051	0.049	0.043	0.052
	BCD	0.013	0.013	0.049	0.041	0.053	0.055	0.047	0.052
	SBR	0.013	0.013	0.040	0.033	0.047	0.046	0.044	0.045

Table 3: H_0, $n = 200$, $\tau = 0.75$

M	A	s/naive	s/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.052	0.052	0.053	0.044	0.044	0.048	0.041	0.042
	WEI	0.012	0.042	0.014	0.043	0.046	0.037	0.039	0.045
	BCD	0.002	0.047	0.002	0.051	0.054	0.055	0.053	0.053
	SBR	0.001	0.026	0.003	0.030	0.035	0.030	0.033	0.035
2	SRS	0.052	0.052	0.066	0.057	0.058	0.053	0.048	0.058
	WEI	0.021	0.045	0.027	0.047	0.052	0.057	0.051	0.054
	BCD	0.013	0.046	0.025	0.051	0.060	0.067	0.061	0.060
	SBR	0.008	0.036	0.012	0.037	0.046	0.046	0.046	0.050
3	SRS	0.058	0.058	0.048	0.054	0.047	0.058	0.054	0.051
	WEI	0.053	0.055	0.041	0.044	0.047	0.047	0.048	0.046
	BCD	0.042	0.043	0.026	0.026	0.033	0.033	0.032	0.034
	SBR	0.048	0.052	0.040	0.036	0.046	0.051	0.043	0.048
4	SRS	0.044	0.044	0.057	0.059	0.062	0.053	0.051	0.065
	WEI	0.034	0.034	0.044	0.029	0.053	0.048	0.044	0.054
	BCD	0.029	0.032	0.040	0.019	0.045	0.047	0.043	0.047
	SBR	0.034	0.037	0.042	0.025	0.051	0.055	0.049	0.051

Table 4: H_0, $n = 400$, $\tau = 0.25$

M	A	s/naive	s/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.047	0.047	0.053	0.041	0.039	0.049	0.040	0.040
	WEI	0.009	0.043	0.017	0.041	0.042	0.045	0.044	0.043
	BCD	0.002	0.042	0.003	0.037	0.040	0.035	0.036	0.037
	SBR	0.002	0.043	0.004	0.034	0.034	0.036	0.032	0.030
2	SRS	0.046	0.046	0.056	0.059	0.059	0.055	0.057	0.059
	WEI	0.035	0.046	0.046	0.056	0.062	0.065	0.061	0.060
	BCD	0.030	0.044	0.037	0.055	0.065	0.060	0.060	0.057
	SBR	0.026	0.049	0.042	0.058	0.067	0.063	0.062	0.066
3	SRS	0.044	0.044	0.039	0.041	0.042	0.042	0.041	0.043
	WEI	0.042	0.045	0.048	0.041	0.048	0.051	0.046	0.049
	BCD	0.039	0.040	0.041	0.040	0.044	0.046	0.047	0.048
	SBR	0.048	0.051	0.046	0.048	0.052	0.056	0.056	0.055
4	SRS	0.056	0.056	0.039	0.042	0.041	0.041	0.043	0.042
	WEI	0.052	0.055	0.038	0.034	0.045	0.042	0.044	0.044
	BCD	0.054	0.058	0.040	0.026	0.045	0.044	0.045	0.043
	SBR	0.061	0.068	0.049	0.027	0.047	0.054	0.055	0.051

Table 5: H_0, $n = 400$, $\tau = 0.5$

M	A	s/naive	s/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.042	0.042	0.054	0.046	0.040	0.046	0.050	0.041
	WEI	0.010	0.049	0.008	0.047	0.047	0.046	0.043	0.042
	BCD	0.003	0.045	0.002	0.043	0.043	0.035	0.039	0.040
	SBR	0.002	0.046	0.000	0.035	0.037	0.036	0.036	0.037
2	SRS	0.050	0.050	0.055	0.049	0.047	0.051	0.052	0.050
	WEI	0.018	0.048	0.025	0.041	0.046	0.045	0.048	0.045
	BCD	0.011	0.042	0.011	0.041	0.046	0.045	0.046	0.043
	SBR	0.017	0.051	0.014	0.042	0.050	0.053	0.047	0.050
3	SRS	0.012	0.012	0.043	0.046	0.048	0.046	0.050	0.050
	WEI	0.014	0.016	0.057	0.055	0.060	0.055	0.058	0.057
	BCD	0.013	0.013	0.055	0.059	0.061	0.051	0.053	0.052
	SBR	0.006	0.006	0.040	0.040	0.039	0.038	0.039	0.038
4	SRS	0.019	0.019	0.056	0.052	0.064	0.056	0.051	0.061
	WEI	0.018	0.018	0.060	0.046	0.065	0.064	0.062	0.066
	BCD	0.015	0.015	0.057	0.046	0.066	0.063	0.059	0.067
	SBR	0.021	0.021	0.057	0.043	0.060	0.062	0.062	0.062

Table 6: H_0, $n = 400$, $\tau = 0.75$

M	A	s/naive	s/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.051	0.051	0.056	0.055	0.056	0.052	0.055	0.054
	WEI	0.007	0.041	0.014	0.055	0.053	0.051	0.050	0.051
	BCD	0.006	0.038	0.004	0.046	0.048	0.041	0.042	0.046
	SBR	0.004	0.033	0.002	0.044	0.043	0.042	0.043	0.042
2	SRS	0.048	0.048	0.073	0.055	0.061	0.060	0.057	0.059
	WEI	0.020	0.039	0.024	0.046	0.053	0.048	0.051	0.053
	BCD	0.012	0.048	0.020	0.050	0.051	0.057	0.055	0.051
	SBR	0.011	0.047	0.014	0.046	0.052	0.050	0.052	0.052
3	SRS	0.054	0.054	0.050	0.045	0.052	0.049	0.044	0.052
	WEI	0.053	0.055	0.049	0.047	0.053	0.050	0.049	0.054
	BCD	0.059	0.063	0.038	0.041	0.045	0.044	0.043	0.043
	SBR	0.049	0.051	0.042	0.044	0.043	0.049	0.049	0.049
4	SRS	0.054	0.054	0.057	0.053	0.063	0.055	0.056	0.063
	WEI	0.047	0.051	0.055	0.043	0.064	0.055	0.061	0.059
	BCD	0.049	0.051	0.054	0.033	0.063	0.062	0.056	0.063
	SBR	0.046	0.048	0.047	0.026	0.051	0.057	0.056	0.053

S.F.3 QTE, H_1, $\pi = 0.5$

Table 7: H_1, $n = 200$, $\tau = 0.25$

M	A	s/naive	s/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.191	0.191	0.203	0.354	0.356	0.205	0.340	0.342
	WEI	0.126	0.257	0.147	0.359	0.358	0.279	0.345	0.350
	BCD	0.105	0.372	0.122	0.379	0.375	0.361	0.369	0.365
	SBR	0.099	0.400	0.114	0.378	0.382	0.411	0.375	0.368
2	SRS	0.284	0.284	0.315	0.352	0.376	0.319	0.345	0.378
	WEI	0.270	0.319	0.314	0.356	0.364	0.359	0.363	0.369
	BCD	0.282	0.333	0.304	0.361	0.375	0.390	0.385	0.383
	SBR	0.290	0.346	0.296	0.335	0.361	0.387	0.358	0.356
3	SRS	0.712	0.712	0.694	0.688	0.698	0.704	0.677	0.686
	WEI	0.701	0.707	0.678	0.685	0.680	0.699	0.687	0.674
	BCD	0.712	0.720	0.673	0.686	0.695	0.699	0.698	0.698
	SBR	0.672	0.684	0.659	0.639	0.647	0.673	0.647	0.638
4	SRS	0.166	0.166	0.124	0.112	0.132	0.135	0.131	0.128
	WEI	0.166	0.170	0.126	0.098	0.125	0.144	0.139	0.133
	BCD	0.165	0.176	0.126	0.094	0.155	0.157	0.145	0.157
	SBR	0.167	0.175	0.122	0.088	0.139	0.145	0.133	0.140

Table 8: H_1, $n = 200$, $\tau = 0.5$

M	A	s/naive	s/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.183	0.183	0.193	0.443	0.441	0.200	0.431	0.429
	WEI	0.116	0.295	0.138	0.442	0.447	0.298	0.437	0.436
	BCD	0.072	0.472	0.095	0.450	0.453	0.434	0.446	0.448
	SBR	0.085	0.485	0.099	0.463	0.460	0.457	0.453	0.448
2	SRS	0.267	0.267	0.256	0.359	0.366	0.265	0.358	0.371
	WEI	0.248	0.346	0.247	0.358	0.394	0.346	0.378	0.389
	BCD	0.229	0.402	0.233	0.358	0.396	0.388	0.395	0.392
	SBR	0.232	0.404	0.234	0.365	0.392	0.399	0.401	0.391
3	SRS	0.797	0.797	0.904	0.897	0.916	0.902	0.897	0.913
	WEI	0.802	0.807	0.907	0.903	0.909	0.913	0.902	0.906
	BCD	0.796	0.804	0.902	0.910	0.911	0.908	0.911	0.906
	SBR	0.771	0.774	0.897	0.896	0.901	0.899	0.894	0.899
4	SRS	0.176	0.176	0.312	0.269	0.317	0.316	0.297	0.316
	WEI	0.171	0.175	0.289	0.255	0.307	0.309	0.297	0.298
	BCD	0.169	0.174	0.299	0.262	0.313	0.329	0.311	0.316
	SBR	0.163	0.165	0.283	0.255	0.304	0.302	0.298	0.298

Table 9: H_1, $n = 200$, $\tau = 0.75$

M	A	s/naive	s/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.198	0.198	0.215	0.362	0.358	0.216	0.353	0.355
	WEI	0.143	0.293	0.153	0.361	0.368	0.315	0.362	0.364
	BCD	0.108	0.377	0.131	0.356	0.360	0.355	0.353	0.353
	SBR	0.079	0.386	0.105	0.397	0.396	0.381	0.403	0.386
2	SRS	0.268	0.268	0.315	0.386	0.439	0.322	0.391	0.434
	WEI	0.238	0.339	0.285	0.396	0.430	0.390	0.417	0.428
	BCD	0.209	0.407	0.263	0.398	0.428	0.425	0.428	0.418
	SBR	0.206	0.427	0.267	0.439	0.455	0.450	0.465	0.456
3	SRS	0.698	0.698	0.607	0.594	0.619	0.634	0.609	0.622
	WEI	0.668	0.673	0.607	0.606	0.616	0.631	0.623	0.624
	BCD	0.690	0.698	0.607	0.612	0.616	0.635	0.618	0.621
	SBR	0.669	0.675	0.596	0.614	0.633	0.617	0.631	0.630
4	SRS	0.163	0.163	0.158	0.122	0.167	0.173	0.140	0.169
	WEI	0.144	0.152	0.152	0.105	0.175	0.169	0.152	0.178
	BCD	0.133	0.138	0.151	0.085	0.170	0.177	0.173	0.172
	SBR	0.146	0.154	0.143	0.090	0.175	0.171	0.177	0.180

Table 10: H_1, $n = 400$, $\tau = 0.25$

M	A	s/naive	s/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.206	0.206	0.229	0.403	0.417	0.231	0.401	0.405
	WEI	0.163	0.332	0.173	0.408	0.413	0.337	0.408	0.413
	BCD	0.121	0.430	0.143	0.420	0.422	0.421	0.419	0.413
	SBR	0.128	0.451	0.144	0.428	0.429	0.458	0.426	0.423
2	SRS	0.312	0.312	0.345	0.422	0.415	0.351	0.416	0.416
	WEI	0.312	0.352	0.332	0.405	0.424	0.378	0.408	0.426
	BCD	0.299	0.378	0.333	0.392	0.405	0.403	0.415	0.413
	SBR	0.330	0.389	0.345	0.401	0.407	0.426	0.410	0.406
3	SRS	0.763	0.763	0.734	0.730	0.740	0.738	0.732	0.738
	WEI	0.763	0.764	0.739	0.739	0.748	0.744	0.746	0.746
	BCD	0.781	0.783	0.760	0.760	0.768	0.772	0.774	0.767
	SBR	0.766	0.773	0.745	0.739	0.744	0.763	0.751	0.744
4	SRS	0.177	0.177	0.129	0.108	0.136	0.127	0.121	0.133
	WEI	0.170	0.176	0.129	0.096	0.139	0.139	0.131	0.143
	BCD	0.178	0.185	0.132	0.089	0.141	0.141	0.139	0.138
	SBR	0.180	0.186	0.129	0.102	0.134	0.147	0.135	0.133

Table 11: H_1, $n = 400$, $\tau = 0.5$

M	A	s/naive	s/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.218	0.218	0.232	0.504	0.502	0.235	0.497	0.502
	WEI	0.147	0.356	0.160	0.503	0.503	0.350	0.498	0.507
	BCD	0.089	0.526	0.117	0.498	0.502	0.493	0.495	0.496
	SBR	0.089	0.550	0.109	0.520	0.518	0.524	0.526	0.519
2	SRS	0.301	0.301	0.309	0.402	0.426	0.306	0.413	0.423
	WEI	0.287	0.387	0.281	0.402	0.418	0.372	0.411	0.420
	BCD	0.268	0.451	0.262	0.400	0.443	0.434	0.434	0.441
	SBR	0.260	0.433	0.252	0.403	0.421	0.418	0.431	0.420
3	SRS	0.897	0.897	0.956	0.957	0.956	0.957	0.956	0.957
	WEI	0.892	0.892	0.954	0.944	0.948	0.951	0.942	0.948
	BCD	0.887	0.889	0.952	0.949	0.954	0.957	0.954	0.956
	SBR	0.900	0.902	0.954	0.954	0.954	0.958	0.962	0.957
4	SRS	0.234	0.234	0.345	0.317	0.351	0.353	0.339	0.343
	WEI	0.222	0.224	0.336	0.326	0.352	0.352	0.335	0.358
	BCD	0.226	0.230	0.346	0.321	0.349	0.368	0.359	0.365
	SBR	0.238	0.242	0.369	0.350	0.380	0.379	0.374	0.377

Table 12: H_1, $n = 400$, $\tau = 0.75$

M	A	s/naive	s/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.218	0.218	0.237	0.430	0.435	0.242	0.438	0.435
	WEI	0.163	0.321	0.176	0.441	0.437	0.344	0.433	0.432
	BCD	0.136	0.422	0.152	0.421	0.420	0.417	0.417	0.416
	SBR	0.103	0.446	0.124	0.459	0.459	0.448	0.463	0.461
2	SRS	0.300	0.300	0.337	0.445	0.479	0.335	0.449	0.479
	WEI	0.258	0.369	0.313	0.446	0.465	0.414	0.453	0.463
	BCD	0.247	0.462	0.295	0.451	0.476	0.483	0.481	0.477
	SBR	0.227	0.444	0.276	0.472	0.490	0.471	0.496	0.492
3	SRS	0.763	0.763	0.710	0.702	0.707	0.712	0.701	0.715
	WEI	0.773	0.776	0.696	0.701	0.700	0.720	0.709	0.706
	BCD	0.753	0.755	0.705	0.716	0.720	0.720	0.717	0.726
	SBR	0.746	0.750	0.684	0.699	0.705	0.692	0.709	0.708
4	SRS	0.209	0.209	0.199	0.140	0.221	0.208	0.149	0.221
	WEI	0.201	0.208	0.191	0.110	0.203	0.206	0.178	0.204
	BCD	0.195	0.200	0.199	0.121	0.213	0.224	0.213	0.220
	SBR	0.198	0.203	0.198	0.114	0.229	0.214	0.230	0.225

S.F.4 QTE, H_0, $\pi = 0.7$

Table 13: H_0, $n = 200$, $\tau = 0.25$

M	A	s/naive	s/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.042	0.042	0.046	0.042	0.036	0.036	0.039	0.039
	SBR	0.002	0.014	0.005	0.053	0.052	0.049	0.050	0.047
2	SRS	0.037	0.037	0.051	0.059	0.057	0.061	0.057	0.064
	SBR	0.032	0.036	0.042	0.046	0.048	0.055	0.055	0.055
3	SRS	0.046	0.046	0.046	0.047	0.039	0.045	0.049	0.043
	SBR	0.040	0.044	0.032	0.031	0.034	0.041	0.037	0.040
4	SRS	0.098	0.098	0.067	0.075	0.069	0.062	0.057	0.066
	SBR	0.057	0.066	0.043	0.016	0.062	0.061	0.066	0.064

Table 14: H_0, $n = 200$, $\tau = 0.5$

M	A	s/naive	s/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.048	0.048	0.052	0.045	0.047	0.034	0.040	0.044
	SBR	0.001	0.007	0.002	0.039	0.040	0.044	0.038	0.037
2	SRS	0.057	0.057	0.065	0.051	0.058	0.050	0.051	0.053
	SBR	0.022	0.034	0.021	0.053	0.053	0.050	0.059	0.053
3	SRS	0.016	0.016	0.052	0.046	0.054	0.051	0.048	0.053
	SBR	0.004	0.005	0.039	0.038	0.048	0.045	0.046	0.048
4	SRS	0.009	0.009	0.046	0.037	0.049	0.046	0.045	0.051
	SBR	0.004	0.005	0.036	0.016	0.052	0.049	0.043	0.046

Table 15: H_0, $n = 200$, $\tau = 0.75$

M	A	s/naive	s/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.052	0.052	0.057	0.045	0.049	0.044	0.040	0.043
	SBR	0.002	0.008	0.004	0.033	0.034	0.036	0.036	0.036
2	SRS	0.042	0.042	0.061	0.055	0.067	0.047	0.055	0.068
	SBR	0.006	0.014	0.009	0.029	0.037	0.042	0.039	0.040
3	SRS	0.056	0.056	0.043	0.038	0.054	0.048	0.046	0.054
	SBR	0.055	0.057	0.048	0.042	0.050	0.053	0.052	0.052
4	SRS	0.019	0.019	0.038	0.032	0.046	0.045	0.042	0.042
	SBR	0.022	0.022	0.044	0.028	0.045	0.044	0.038	0.042

Table 16: H_0, $n = 400$, $\tau = 0.25$

M	A	s/naive	s/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.044	0.044	0.054	0.039	0.041	0.038	0.040	0.042
	SBR	0.003	0.015	0.003	0.051	0.052	0.043	0.046	0.046
2	SRS	0.034	0.034	0.057	0.058	0.054	0.062	0.058	0.053
	SBR	0.031	0.034	0.040	0.044	0.049	0.051	0.051	0.051
3	SRS	0.037	0.037	0.029	0.034	0.036	0.033	0.033	0.039
	SBR	0.045	0.049	0.037	0.037	0.042	0.044	0.040	0.041
4	SRS	0.073	0.073	0.044	0.054	0.046	0.045	0.048	0.041
	SBR	0.065	0.076	0.036	0.014	0.060	0.058	0.062	0.060

Table 17: H_0, $n = 400$, $\tau = 0.5$

M	A	s/naive	s/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.044	0.044	0.051	0.037	0.039	0.048	0.036	0.037
	SBR	0.001	0.002	0.000	0.035	0.039	0.035	0.040	0.040
2	SRS	0.062	0.062	0.062	0.049	0.049	0.059	0.041	0.048
	SBR	0.015	0.029	0.015	0.034	0.040	0.040	0.042	0.037
3	SRS	0.007	0.007	0.039	0.036	0.042	0.042	0.042	0.047
	SBR	0.006	0.006	0.035	0.037	0.036	0.037	0.041	0.037
4	SRS	0.013	0.013	0.046	0.029	0.061	0.053	0.035	0.054
	SBR	0.009	0.010	0.033	0.025	0.056	0.054	0.052	0.050

Table 18: H_0, $n = 400$, $\tau = 0.75$

M	A	s/naive	s/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.049	0.049	0.053	0.046	0.050	0.043	0.048	0.050
	SBR	0.001	0.006	0.002	0.038	0.041	0.037	0.036	0.036
2	SRS	0.050	0.050	0.065	0.050	0.049	0.056	0.052	0.052
	SBR	0.010	0.019	0.015	0.041	0.048	0.042	0.041	0.041
3	SRS	0.044	0.044	0.031	0.042	0.039	0.032	0.038	0.039
	SBR	0.057	0.059	0.040	0.036	0.044	0.043	0.043	0.043
4	SRS	0.034	0.034	0.051	0.046	0.049	0.051	0.046	0.051
	SBR	0.028	0.028	0.044	0.040	0.045	0.045	0.045	0.046

S.F.5 QTE, H_1, $\pi = 0.7$

Table 19: H_1, $n = 200$, $\tau = 0.25$

M	A	s/naive	s/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.152	0.152	0.176	0.359	0.313	0.187	0.343	0.339
	SBR	0.065	0.186	0.100	0.346	0.336	0.357	0.341	0.338
2	SRS	0.314	0.314	0.334	0.361	0.325	0.347	0.367	0.365
	SBR	0.309	0.334	0.336	0.355	0.368	0.383	0.375	0.376
3	SRS	0.704	0.704	0.671	0.665	0.626	0.685	0.663	0.691
	SBR	0.697	0.716	0.663	0.671	0.669	0.702	0.686	0.688
4	SRS	0.136	0.136	0.097	0.094	0.129	0.106	0.093	0.122
	SBR	0.116	0.127	0.081	0.050	0.103	0.107	0.105	0.106

Table 20: H_1, $n = 200$, $\tau = 0.5$

M	A	s/naive	s/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.170	0.170	0.172	0.411	0.425	0.167	0.407	0.406
	SBR	0.043	0.212	0.060	0.445	0.455	0.457	0.435	0.434
2	SRS	0.287	0.287	0.280	0.371	0.364	0.275	0.374	0.360
	SBR	0.258	0.327	0.236	0.367	0.387	0.372	0.383	0.381
3	SRS	0.771	0.771	0.891	0.882	0.903	0.895	0.883	0.894
	SBR	0.760	0.769	0.892	0.896	0.911	0.901	0.904	0.900
4	SRS	0.145	0.145	0.265	0.218	0.305	0.264	0.241	0.301
	SBR	0.128	0.136	0.235	0.177	0.288	0.290	0.284	0.287

Table 21: H_1, $n = 200$, $\tau = 0.75$

M	A	s/naive	s/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.181	0.181	0.183	0.342	0.340	0.188	0.340	0.338
	SBR	0.072	0.175	0.076	0.353	0.364	0.342	0.357	0.357
2	SRS	0.279	0.279	0.321	0.404	0.427	0.341	0.400	0.427
	SBR	0.243	0.341	0.293	0.430	0.451	0.430	0.454	0.435
3	SRS	0.662	0.662	0.586	0.559	0.599	0.605	0.569	0.592
	SBR	0.631	0.639	0.572	0.564	0.597	0.594	0.601	0.598
4	SRS	0.150	0.150	0.201	0.164	0.199	0.208	0.189	0.211
	SBR	0.143	0.145	0.193	0.166	0.206	0.206	0.208	0.205

Table 22: H_1, $n = 400$, $\tau = 0.25$

M	A	s/naive	s/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.181	0.181	0.192	0.351	0.354	0.202	0.346	0.351
	SBR	0.083	0.233	0.113	0.392	0.392	0.407	0.394	0.392
2	SRS	0.362	0.362	0.406	0.403	0.415	0.408	0.415	0.424
	SBR	0.350	0.381	0.388	0.412	0.426	0.426	0.422	0.419
3	SRS	0.781	0.781	0.743	0.751	0.758	0.746	0.750	0.759
	SBR	0.791	0.797	0.752	0.765	0.777	0.781	0.778	0.779
4	SRS	0.160	0.160	0.082	0.072	0.112	0.097	0.095	0.116
	SBR	0.133	0.154	0.091	0.044	0.119	0.119	0.121	0.120

Table 23: H_1, $n = 400$, $\tau = 0.5$

M	A	s/naive	s/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.184	0.184	0.187	0.468	0.479	0.194	0.460	0.466
	SBR	0.042	0.220	0.059	0.486	0.498	0.505	0.480	0.482
2	SRS	0.322	0.322	0.298	0.405	0.404	0.303	0.412	0.400
	SBR	0.262	0.342	0.237	0.376	0.399	0.385	0.389	0.389
3	SRS	0.867	0.867	0.939	0.930	0.933	0.941	0.932	0.936
	SBR	0.883	0.888	0.948	0.952	0.952	0.955	0.952	0.952
4	SRS	0.209	0.209	0.327	0.275	0.354	0.341	0.308	0.351
	SBR	0.194	0.217	0.310	0.256	0.365	0.364	0.359	0.356

Table 24: H_1, $n = 400$, $\tau = 0.75$

M	A	s/naive	s/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.217	0.217	0.224	0.411	0.409	0.219	0.411	0.408
	SBR	0.103	0.246	0.107	0.419	0.418	0.400	0.421	0.420
2	SRS	0.335	0.335	0.378	0.485	0.505	0.384	0.468	0.501
	SBR	0.278	0.384	0.329	0.479	0.500	0.487	0.504	0.493
3	SRS	0.708	0.708	0.661	0.628	0.665	0.665	0.629	0.672
	SBR	0.705	0.706	0.652	0.631	0.665	0.673	0.672	0.673
4	SRS	0.205	0.205	0.226	0.221	0.245	0.234	0.234	0.240
	SBR	0.205	0.205	0.249	0.209	0.248	0.258	0.256	0.258

S.F.6 ATE, $\pi = 0.5$

Table 25: H_0, $n = 200$, $\pi = 0.5$

M	A	s/naive	s/adj	sfe/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.059	0.057	0.051	0.061	0.055	0.057	0.053	0.048	0.049
	WEI	0.006	0.048	0.062	0.004	0.068	0.068	0.051	0.065	0.065
	BCD	0.001	0.089	0.056	0.000	0.058	0.058	0.071	0.056	0.056
	SBR	0.000	0.067	0.061	0.000	0.064	0.064	0.059	0.061	0.061
2	SRS	0.062	0.061	0.061	0.061	0.059	0.062	0.060	0.057	0.059
	WEI	0.027	0.060	0.050	0.029	0.046	0.054	0.057	0.052	0.053
	BCD	0.014	0.058	0.053	0.016	0.053	0.052	0.052	0.052	0.049
	SBR	0.006	0.045	0.044	0.006	0.045	0.045	0.045	0.045	0.045
3	SRS	0.057	0.056	0.068	0.055	0.061	0.061	0.056	0.064	0.065
	WEI	0.049	0.050	0.057	0.052	0.057	0.056	0.048	0.053	0.053
	BCD	0.057	0.058	0.057	0.057	0.063	0.063	0.057	0.056	0.057
	SBR	0.055	0.058	0.056	0.057	0.060	0.061	0.055	0.055	0.055
4	SRS	0.066	0.067	0.077	0.068	0.069	0.063	0.063	0.070	0.063
	WEI	0.065	0.067	0.070	0.066	0.067	0.068	0.069	0.067	0.070
	BCD	0.068	0.068	0.067	0.065	0.061	0.068	0.065	0.065	0.065
	SBR	0.055	0.055	0.055	0.057	0.057	0.058	0.057	0.057	0.057

Table 26: H_1, $n = 200$, $\pi = 0.5$

M	A	s/naive	s/adj	sfe/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.387	0.385	0.948	0.391	0.946	0.946	0.386	0.944	0.942
	WEI	0.330	0.680	0.944	0.334	0.941	0.940	0.691	0.942	0.941
	BCD	0.275	0.917	0.940	0.272	0.943	0.943	0.884	0.942	0.942
	SBR	0.280	0.942	0.951	0.285	0.950	0.950	0.937	0.945	0.945
2	SRS	0.533	0.532	0.750	0.538	0.746	0.758	0.541	0.746	0.753
	WEI	0.532	0.668	0.748	0.533	0.742	0.750	0.675	0.743	0.749
	BCD	0.541	0.748	0.752	0.544	0.751	0.755	0.733	0.751	0.752
	SBR	0.544	0.774	0.779	0.551	0.772	0.781	0.769	0.775	0.775
3	SRS	0.770	0.769	0.767	0.773	0.768	0.775	0.769	0.754	0.760
	WEI	0.760	0.766	0.763	0.759	0.759	0.768	0.765	0.763	0.761
	BCD	0.767	0.772	0.769	0.762	0.771	0.769	0.772	0.765	0.765
	SBR	0.757	0.762	0.761	0.758	0.770	0.767	0.761	0.764	0.764
4	SRS	0.181	0.182	0.181	0.182	0.171	0.184	0.181	0.180	0.186
	WEI	0.180	0.183	0.182	0.184	0.180	0.184	0.184	0.178	0.179
	BCD	0.170	0.175	0.174	0.177	0.177	0.181	0.182	0.183	0.182
	SBR	0.177	0.178	0.179	0.184	0.180	0.186	0.179	0.178	0.178

Table 27: H_0, $n = 400$, $\pi = 0.5$

M	A	s/naive	s/adj	sfe/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.063	0.061	0.042	0.063	0.043	0.045	0.055	0.042	0.042
	WEI	0.005	0.050	0.050	0.006	0.052	0.052	0.052	0.050	0.050
	BCD	0.000	0.067	0.052	0.000	0.059	0.059	0.051	0.059	0.059
	SBR	0.000	0.059	0.058	0.000	0.057	0.057	0.063	0.060	0.060
2	SRS	0.061	0.057	0.055	0.058	0.055	0.054	0.061	0.054	0.051
	WEI	0.018	0.051	0.064	0.019	0.063	0.064	0.052	0.064	0.064
	BCD	0.009	0.045	0.046	0.006	0.046	0.047	0.043	0.049	0.049
	SBR	0.014	0.062	0.060	0.016	0.065	0.065	0.063	0.063	0.063
3	SRS	0.050	0.049	0.050	0.050	0.049	0.051	0.052	0.048	0.048
	WEI	0.046	0.047	0.049	0.047	0.046	0.047	0.048	0.047	0.046
	BCD	0.049	0.049	0.049	0.049	0.050	0.050	0.050	0.050	0.050
	SBR	0.055	0.056	0.056	0.059	0.058	0.059	0.055	0.056	0.056
4	SRS	0.057	0.057	0.055	0.056	0.056	0.059	0.054	0.051	0.056
	WEI	0.051	0.051	0.053	0.052	0.054	0.054	0.051	0.051	0.052
	BCD	0.056	0.056	0.056	0.054	0.056	0.056	0.054	0.053	0.053
	SBR	0.056	0.058	0.058	0.055	0.056	0.057	0.057	0.057	0.057

Table 28: H_1, $n = 400$, $\pi = 0.5$

M	A	s/naive	s/adj	sfe/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.422	0.422	0.964	0.416	0.968	0.966	0.415	0.964	0.962
	WEI	0.387	0.732	0.969	0.393	0.969	0.969	0.732	0.967	0.968
	BCD	0.341	0.962	0.971	0.350	0.969	0.968	0.955	0.968	0.968
	SBR	0.357	0.967	0.967	0.368	0.966	0.966	0.967	0.965	0.965
2	SRS	0.572	0.568	0.806	0.579	0.795	0.805	0.568	0.796	0.805
	WEI	0.577	0.723	0.813	0.575	0.814	0.810	0.728	0.811	0.808
	BCD	0.606	0.809	0.813	0.618	0.817	0.821	0.802	0.810	0.810
	SBR	0.601	0.828	0.829	0.603	0.832	0.836	0.830	0.834	0.834
3	SRS	0.804	0.801	0.803	0.798	0.798	0.799	0.804	0.803	0.803
	WEI	0.804	0.804	0.806	0.802	0.800	0.803	0.803	0.803	0.803
	BCD	0.816	0.818	0.820	0.822	0.825	0.825	0.819	0.819	0.819
	SBR	0.821	0.823	0.823	0.816	0.820	0.819	0.822	0.822	0.822
4	SRS	0.228	0.230	0.229	0.225	0.227	0.228	0.234	0.226	0.226
	WEI	0.229	0.230	0.230	0.225	0.223	0.228	0.233	0.235	0.234
	BCD	0.221	0.224	0.225	0.227	0.225	0.231	0.231	0.231	0.233
	SBR	0.224	0.226	0.225	0.224	0.225	0.230	0.235	0.235	0.235

S.F.7 ATE, $\pi = 0.7$

Table 29: H_0, $n = 200$, $\pi = 0.7$

M	A	s/naive	s/adj	sfe/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.050	0.045	0.056	0.051	0.056	0.062	0.046	0.054	0.055
	SBR	0.000	0.004	0.051	0.000	0.061	0.064	0.064	0.060	0.059
2	SRS	0.048	0.055	0.074	0.055	0.049	0.056	0.045	0.049	0.057
	SBR	0.013	0.030	0.041	0.013	0.024	0.051	0.056	0.049	0.051
3	SRS	0.059	0.060	0.066	0.060	0.060	0.064	0.058	0.055	0.064
	SBR	0.051	0.053	0.052	0.053	0.045	0.057	0.056	0.056	0.055
4	SRS	0.057	0.057	0.056	0.058	0.056	0.068	0.054	0.057	0.058
	SBR	0.047	0.050	0.044	0.051	0.037	0.054	0.054	0.055	0.055

Table 30: H_1, $n = 200$, $\pi = 0.7$

M	A	s/naive	s/adj	sfe/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.329	0.328	0.934	0.336	0.943	0.946	0.326	0.941	0.941
	SBR	0.220	0.631	0.938	0.233	0.946	0.949	0.932	0.943	0.943
2	SRS	0.581	0.578	0.687	0.582	0.619	0.756	0.571	0.601	0.758
	SBR	0.598	0.699	0.747	0.599	0.686	0.768	0.752	0.766	0.764
3	SRS	0.773	0.779	0.758	0.769	0.741	0.784	0.773	0.729	0.782
	SBR	0.771	0.773	0.772	0.777	0.763	0.782	0.782	0.780	0.781
4	SRS	0.149	0.154	0.121	0.153	0.140	0.168	0.154	0.141	0.165
	SBR	0.144	0.151	0.129	0.153	0.118	0.175	0.172	0.170	0.169

Table 31: H_0, $n = 400$, $\pi = 0.7$

M	A	s/naive	s/adj	sfe/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.062	0.059	0.065	0.061	0.056	0.056	0.062	0.060	0.061
	SBR	0.000	0.000	0.034	0.000	0.039	0.040	0.045	0.045	0.044
2	SRS	0.052	0.050	0.087	0.054	0.055	0.052	0.050	0.057	0.051
	SBR	0.013	0.029	0.040	0.012	0.027	0.044	0.042	0.044	0.042
3	SRS	0.042	0.041	0.049	0.045	0.043	0.052	0.040	0.040	0.046
	SBR	0.028	0.028	0.031	0.029	0.025	0.032	0.035	0.036	0.034
4	SRS	0.053	0.055	0.043	0.058	0.053	0.058	0.055	0.050	0.056
	SBR	0.050	0.051	0.043	0.051	0.035	0.054	0.055	0.055	0.053

Table 32: H_1, $n = 400$, $\pi = 0.7$

M	A	s/naive	s/adj	sfe/adj	s/W	sfe/W	ipw/W	s/CA	sfe/CA	ipw/CA
1	SRS	0.384	0.380	0.972	0.381	0.971	0.976	0.382	0.970	0.973
	SBR	0.250	0.736	0.970	0.254	0.972	0.972	0.967	0.973	0.974
2	SRS	0.616	0.628	0.753	0.622	0.693	0.796	0.617	0.690	0.795
	SBR	0.659	0.759	0.806	0.665	0.740	0.827	0.817	0.827	0.827
3	SRS	0.818	0.817	0.805	0.812	0.793	0.821	0.816	0.793	0.829
	SBR	0.833	0.838	0.836	0.831	0.824	0.840	0.838	0.839	0.837
4	SRS	0.177	0.172	0.145	0.180	0.162	0.195	0.181	0.171	0.186
	SBR	0.181	0.190	0.164	0.184	0.142	0.202	0.202	0.202	0.200

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