

Supplement to “The empirical content of games with bounded regressors”

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BRENDAN KLINE

Department of Economics, University of Texas at Austin

APPENDIX A: PROOF OF EXISTENCE OF PURE STRATEGY NASH EQUILIBRIUM

Consider the N -player game with discrete action space $S = \{0, 1, \dots, H\}$.

Let $\theta_i = \tilde{\alpha}_i + x_i\beta_{ix} + w\beta_{iw} + \varepsilon_i$. Then $u_i(y'_i, y_{-i}) - u_i(y_i, y_{-i}) = (\theta_i + \Delta_i(\sum_{j \neq i} y_j + y'_i + y_i))(y'_i - y_i)$ and $u_i(y'_i, y'_{-i}) - u_i(y_i, y'_{-i}) = (\theta_i + \Delta_i(\sum_{j \neq i} y'_j + y'_i + y_i))(y'_i - y_i)$. Therefore, $u_i(y'_i, y_{-i}) - u_i(y_i, y_{-i}) - (u_i(y'_i, y'_{-i}) - u_i(y_i, y'_{-i})) = \Delta_i(\sum_{j \neq i} y_j - \sum_{j \neq i} y'_j)(y'_i - y_i)$.

Therefore, if $\Delta_i \leq 0$ (respectively, $\Delta_i \geq 0$), by comparative statics the best response correspondence for agent i is weakly decreasing (respectively, weakly increasing) with respect to $\sum_{j \neq i} y_j$. Therefore, as long as $\Delta_i \geq 0$ for all agents i , or $\Delta_i \leq 0$ for all agents i , by [Dubey, Haimanko, and Zapechelnyuk \(2006\)](#) (among many other results) there is a pure strategy Nash equilibrium.

APPENDIX B: EXTENSION: IDENTIFICATION OF THE DIRECTION OF THE INTERACTION EFFECT WITH INDEPENDENCE

It is possible to *nonparametrically* point identify the sign of the interaction effect using a strategy similar to [de Paula and Tang \(2012\)](#) for incomplete information. The game in normal form is given in Table S1. These utility functions are nonparametric.

Then let

$$\tilde{\Delta} = \begin{cases} 1, & \text{if } P(u_1(1, 1) > u_1(1, 0), u_2(1, 1) > u_2(0, 1)) = 1, \\ 0, & \text{if } P(u_1(1, 1) = u_1(1, 0), u_2(1, 1) = u_2(0, 1)) = 1, \\ -1, & \text{if } P(u_1(1, 1) < u_1(1, 0), u_2(1, 1) < u_2(0, 1)) = 1. \end{cases}$$

This section shows how to identify $\tilde{\Delta}$. It is assumed that one of these cases hold (i.e., it cannot be that the interaction effect is sometimes positive and sometimes negative). Otherwise, the interaction effect (i.e., $u_{1m}(1, 1) - u_{1m}(1, 0)$ and $u_{2m}(1, 1) - u_{2m}(0, 1)$) can have heterogeneity of unrestricted form. For example, in the canonical linear model $u_{im}(1, y_{(-i)m}) = \alpha_i + x_{im}\beta_i + \Delta y_{(-i)m} + \varepsilon_{im}$, it follows that $\text{sgn}(\Delta) = \tilde{\Delta}$.

Brendan Kline: brendan.kline@austin.utexas.edu

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TABLE S1. Nonparametric specification of utility functions.

	0	1
0	(0, 0)	(0, $u_{2m}(0, 1)$)
1	($u_{1m}(1, 0)$, 0)	($u_{1m}(1, 1)$, $u_{2m}(1, 1)$)

THEOREM B.1 (Nonparametric Identification of the Sign of the Interaction Effect With Independence). *Suppose that the model of the interaction is given in normal form in Table S1, and suppose there is pure strategy Nash equilibrium play. Suppose that there is zero probability that any component of $u = (u_1(1, 0), u_1(1, 1), u_2(0, 1), u_2(1, 1))$ equals zero, and $(u_1(1, 0), u_1(1, 1)) \perp (u_2(0, 1), u_2(1, 1))$. Also suppose that $0 < P(y_1, y_2) < 1$ for all $(y_1, y_2) \in \{0, 1\}^2$. Then the following statements hold:*

(1) *If either $\tilde{\Delta} = 0$ or both $P(\text{sgn}(u_1(1, 0)) \neq \text{sgn}(u_1(1, 1))) > 0$ and $P(\text{sgn}(u_2(0, 1)) \neq \text{sgn}(u_2(1, 1))) > 0$, then $\tilde{\Delta} = \text{sgn} \log\left(\frac{P(y_1=1, y_2=1)}{P(y_1=1)P(y_2=1)}\right)$.*

(2) *In general, if $\tilde{\Delta} \leq 0$, then $\log\left(\frac{P(y_1=1, y_2=1)}{P(y_1=1)P(y_2=1)}\right) \leq 0$, and if $\tilde{\Delta} \geq 0$, then $\log\left(\frac{P(y_1=1, y_2=1)}{P(y_1=1)P(y_2=1)}\right) \geq 0$. Also, if $\log\left(\frac{P(y_1=1, y_2=1)}{P(y_1=1)P(y_2=1)}\right) < 0$, then $\tilde{\Delta} < 0$, and if $\log\left(\frac{P(y_1=1, y_2=1)}{P(y_1=1)P(y_2=1)}\right) > 0$, then $\tilde{\Delta} > 0$.*

REMARK B.1 (Intuition for Identification Strategy). The equality $\tilde{\Delta} = 1$ induces a positive correlation between y_1 and y_2 , since $y_j = 1$ increases the probability that action 1 is the utility maximizing action of agent i ; conversely, $\tilde{\Delta} = -1$ induces a negative correlation between y_1 and y_2 , since $y_j = 1$ decreases the probability that action 1 is the utility maximizing action of agent i . This can be captured by the pointwise mutual information statistic $\log\left(\frac{P(y_1=1, y_2=1)}{P(y_1=1)P(y_2=1)}\right)$.

REMARK B.2 (Condition on Sign of Utility Functions). The condition in part (1) of Theorem B.1 rules out a situation in which there is a nonzero interaction effect (i.e., $\tilde{\Delta} \neq 0$), but effectively no strategic interaction because agents have utility functions that are always positive (or always negative) no matter what the other agent does. In such a case, there is effectively a zero interaction effect. In the canonical linear model, for example with $\tilde{\Delta} < 0$, this condition is equivalent to $P(\alpha_i + x_i\beta_i + \Delta_i < -\varepsilon_i < \alpha_i + x_i\beta_i) > 0$, which would be implied by unobservables with support on the real line.

REMARK B.3 (Equilibrium Existence). A pure strategy Nash equilibrium exists (with probability 1) by an argument similar to [Kline and Tamer \(2012\)](#).

REMARK B.4 (Explanatory Variables). It is possible to do the analysis conditional on explanatory variables x , which may increase the credibility of the independence assumption: the assumption is that the unobservables are independent across agents.

REMARK B.5 (Comparison to de Paula and Tang (2012)). de Paula and Tang (2012) show under their (similar) assumptions in the context of incomplete information games that if $\log\left(\frac{P(y_1=1, y_2=1)}{P(y_1=1)P(y_2=1)}\right) > 0$, then $\tilde{\Delta} = 1$, and if $\log\left(\frac{P(y_1=1, y_2=1)}{P(y_1=1)P(y_2=1)}\right) < 0$, then $\tilde{\Delta} = -1$.¹ As long as $\log\left(\frac{P(y_1=1, y_2=1)}{P(y_1=1)P(y_2=1)}\right) \neq 0$ the results overlap, implying that the use of $\log\left(\frac{P(y_1=1, y_2=1)}{P(y_1=1)P(y_2=1)}\right)$ as a statistic for $\tilde{\Delta}$ is partially robust to different conditions on information (i.e., incomplete information versus complete information). But the proofs are quite different, because they apply to different conditions on information.

But also there are some important differences in the results; in particular, the de Paula and Tang (2012) result is silent about $\tilde{\Delta}$ if $\log\left(\frac{P(y_1=1, y_2=1)}{P(y_1=1)P(y_2=1)}\right) = 0$. They also show that when there is incomplete information, $\log\left(\frac{P(y_1=1, y_2=1)}{P(y_1=1)P(y_2=1)}\right) \neq 0$ if and only if there are multiple Bayesian Nash equilibria that are played in the data generating process. In contrast, with complete information, these results show that $\log\left(\frac{P(y_1=1, y_2=1)}{P(y_1=1)P(y_2=1)}\right) = 0$ is equivalent to $\tilde{\Delta} = 0$.

B.1 Nonparametric estimation of the sign of the interaction effect

Use the notation that $P_M(\cdot)$ is the sample distribution of M independent markets.

THEOREM B.2 (Nonparametric Estimation of $\tilde{\Delta}$). *Under the same conditions as part (1) of Theorem B.1,*

$$\begin{aligned} \hat{\Delta}_M &= \operatorname{sgn}\left(1 \left\{ M^{1/4} \left| \frac{P_M(y_1 = 1, y_2 = 1)}{P_M(y_1 = 1)P_M(y_2 = 1)} - 1 \right| \geq 1 \right\} \right) \\ &\quad \times \log\left(\frac{P_M(y_1 = 1, y_2 = 1)}{P_M(y_1 = 1)P_M(y_2 = 1)}\right) \\ &\rightarrow^{a.s.} \tilde{\Delta}. \end{aligned}$$

The indicator is used because when $\tilde{\Delta} = 0$, generally $\operatorname{sgn}\left(\log\left(\frac{P_M(y_1=1, y_2=1)}{P_M(y_1=1)P_M(y_2=1)}\right)\right) \neq 0$. This is similar to tests for moment inequalities in Andrews and Soares (2010) and Kline (2011), where it is necessary to know which moment inequalities bind. Since the support of $\hat{\Delta}$ is finite, with probability 1, in large enough samples, $\hat{\Delta} = \tilde{\Delta}$. So the rate of convergence is arbitrarily fast.

APPENDIX C: PROOFS OF RESULTS IN SUPPLEMENT

PROOF OF THEOREM B.1. The proof is equivalent to showing that $\tilde{\Delta} = \operatorname{sgn}\left(\frac{P(y_1=1, y_2=1)}{P(y_1=1)P(y_2=1)} - 1\right)$. Suppose that $\tilde{\Delta} = -1$. Since $P(y_1 = 1, y_2 = 1) > 0$, it follows that $P(u_1(1, 1) \geq 0, u_2(1, 1) \geq 0) > 0$.² So $P(u_1(1, 1) \geq 0) > 0$ and $P(u_2(1, 1) \geq 0) > 0$. Similarly,

¹de Paula and Tang (2012) do not literally give this result, but the equivalence is evident after translating the notation, and some algebra.

²It holds that $P(y_1 = 1, y_2 = 1) = P(u_1(1, 1) \geq 0, u_2(1, 1) \geq 0)$ since if $(1, 1)$ is the pure strategy Nash equilibrium, then $u_1(1, 1) \geq 0$ and $u_2(1, 1) \geq 0$. Conversely, if $u_1(1, 1) > 0$ and $u_2(1, 1) > 0$, then $(1, 1)$ is the unique pure strategy Nash equilibrium. The event that $u_1(1, 1) = 0$ or $u_2(1, 1) = 0$ has zero probability by assumption.

since $P(y_1 = 0, y_2 = 0) > 0$, $P(u_1(1, 0) \leq 0) > 0$ and $P(u_2(0, 1) \leq 0) > 0$. By the assumptions in part (1), $P(u_1(1, 0) > 0 > u_1(1, 1)) > 0$ and $P(u_2(0, 1) > 0 > u_2(1, 1)) > 0$. Since $u_1 \perp u_2$, this implies $P(u_1(1, 0) > 0 > u_1(1, 1), u_2(0, 1) < 0) > 0$.

Then $P(y_1 = 1 | y_2 = 1) = \frac{P(y_1=1, y_2=1)}{P(y_2=1)} = \frac{P(u_1(1,1) \geq 0, u_2(1,1) \geq 0)}{P(y_2=1)}$. Further, $P(y_2 = 1) \geq P(u_2(1, 1) \geq 0)$, since whenever $u_2(1, 1) > 0$, $y_2 = 1$ is a strictly dominant strategy, and $P(u_2(1, 1) = 0) = 0$. Therefore, $P(y_1 = 1 | y_2 = 1) \leq \frac{P(u_1(1,1) \geq 0, u_2(1,1) \geq 0)}{P(u_2(1,1) \geq 0)} = P(u_1(1, 1) \geq 0) \leq P(y_1 = 1)$ since $u_1 \perp u_2$. This implies $\frac{P(y_1=1, y_2=1)}{P(y_1=1)P(y_2=1)} \leq 1$. The inequality is strict under the assumptions in part (1), because $P(u_1(1, 0) > 0 > u_1(1, 1), u_2(0, 1) < 0) > 0$, which also results in the Nash equilibrium outcome $y_1 = 1$, so $P(y_1 = 1) > P(u_1(1, 1) \geq 0)$.

Suppose that $\tilde{\Delta} = 1$. By symmetric arguments, $\frac{P(y_1=0, y_2=1)}{P(y_1=0)P(y_2=1)} < (\leq) 1$, where the inequality is strict or weak depending on whether the assumption in part (1) is maintained. This is equivalent to $P(y_1 = 0, y_2 = 1) < (\leq) P(y_2 = 1) - P(y_1 = 1)P(y_2 = 1)$, which is equivalent to $\frac{P(y_1=1, y_2=1)}{P(y_1=1)P(y_2=1)} > (\geq) 1$.

Suppose that $\tilde{\Delta} = 0$. Then $P(y_1 = 1, y_2 = 1) = P(u_1(1, 0) \geq 0, u_2(0, 1) \geq 0)$, and $P(y_1 = 1) = P(u_1(1, 0) \geq 0)$ and $P(y_2 = 1) = P(u_2(0, 1) \geq 0)$. So, since $u_1 \perp u_2$, this implies that $\frac{P(y_1=1, y_2=1)}{P(y_1=1)P(y_2=1)} = 1$. \square

PROOF OF THEOREM B.2. By the law of large numbers,

$$\frac{P_M(y_1 = 1, y_2 = 1)}{P_M(y_1 = 1)P_M(y_2 = 1)} \xrightarrow{a.s.} \frac{P(y_1 = 1, y_2 = 1)}{P(y_1 = 1)P(y_2 = 1)}.$$

So if $\tilde{\Delta} \neq 0$, then $\frac{P_M(y_1=1, y_2=1)}{P_M(y_1=1)P_M(y_2=1)} - 1$ converges almost surely to a nonzero number, so the left hand side of the argument in the indicator function converges to $+\infty$ almost surely. Therefore, with probability 1, the indicator function is 1 for sufficiently large sample size. Thus, $\tilde{\Delta} \xrightarrow{a.s.} \tilde{\Delta}$ when $\tilde{\Delta} \neq 0$. Alternatively, supposing that $\tilde{\Delta} = 0$,

$$\begin{aligned} & 1 \left\{ M^{1/4} \left| \frac{P_M(y_1 = 1, y_2 = 1)}{P_M(y_1 = 1)P_M(y_2 = 1)} - 1 \right| \geq 1 \right\} \\ &= 1 \left\{ M^{1/4} |P_M(y_1 = 1, y_2 = 1) - P_M(y_1 = 1)P_M(y_2 = 1)| \right. \\ &\quad \left. \geq P_M(y_1 = 1)P_M(y_2 = 1) \right\} \\ &= 1 \left\{ M^{1/4} |P_M(y_1 = 1, y_2 = 1) - P(y_1 = 1, y_2 = 1) + P(y_1 = 1)P(y_2 = 1) \right. \\ &\quad \left. - P_M(y_1 = 1)P_M(y_2 = 1)| \geq P_M(y_1 = 1)P_M(y_2 = 1) \right\} \\ &= 1 \left\{ M^{1/4} |P_M(y_1 = 1, y_2 = 1) - P(y_1 = 1, y_2 = 1) \right. \\ &\quad \left. + (P(y_1 = 1) - P_M(y_1 = 1))P_M(y_2 = 1) \right. \\ &\quad \left. + (P(y_2 = 1) - P_M(y_2 = 1))P(y_1 = 1) \right| \geq P_M(y_1 = 1)P_M(y_2 = 1) \left. \right\} \\ &\leq 1 \left\{ M^{1/4} |P_M(y_1 = 1, y_2 = 1) - P(y_1 = 1, y_2 = 1)| \right. \\ &\quad \left. + M^{1/4} |P(y_1 = 1) - P_M(y_1 = 1)|P_M(y_2 = 1) \right. \\ &\quad \left. + M^{1/4} |P(y_2 = 1) - P_M(y_2 = 1)|P(y_1 = 1) \geq P_M(y_1 = 1)P_M(y_2 = 1) \right\}. \end{aligned}$$

The second equality follows from $\tilde{\Delta} = 0$ implying that $P(y_1 = 1, y_2 = 1) = P(y_1 = 1)P(y_2 = 1)$. By the Marcinkiewicz and Zygmund (1937) strong law of large numbers, $M^{1/4}(P_M(y_1 = 1, y_2 = 1) - P(y_1 = 1, y_2 = 1)) \rightarrow^{a.s.} 0$, $M^{1/4}(P(y_1 = 1) - P_M(y_1 = 1)) \rightarrow^{a.s.} 0$, and $M^{1/4}(P(y_2 = 1) - P_M(y_2 = 1)) \rightarrow^{a.s.} 0$. Therefore, since also $P_M(y_1 = 1)P_M(y_2 = 1) \rightarrow^{a.s.} P(y_1 = 1)P(y_2 = 1) \neq 0$, the entire indicator function converges almost surely to 0. This means that with probability 1, for sufficiently large sample size, the indicator function is 0. Therefore, $\widehat{\Delta} \rightarrow^{a.s.} \tilde{\Delta}$. \square

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