

# Bond Risk Premia in Consumption-based Models\*

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## Abstract

Gaussian affine term structure models attribute time-varying bond risk premia to changing risk prices driven by the conditional means of the risk factors, while structural models with recursive preferences credit it to stochastic volatility. We reconcile these competing channels by introducing a novel form of stochastic rate of time preference into an otherwise standard model with recursive preferences. Our model is affine and has analytical bond prices making it empirically tractable. We use particle Markov chain Monte Carlo to estimate the model, and find that time variation in bond term premia is predominantly driven by the risk price channel.

**Keywords:** bond risk premia; term structure of interest rates; stochastic rate of time preference; MCMC; particle filter; recursive preferences; stochastic volatility.

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# 1 Introduction

Term premia, risk premia in the bond market, are a key object of interest for central banks. They influence how monetary policy implementation via the short term interest rate gets transmitted into the real economy through borrowing costs at longer maturities, and ultimately determines its effectiveness. Policy speeches – made by the Fed Chairs Yellen (2014), Bernanke (2006) and Greenspan (2005), for example – call for the importance of understanding how and why term premia fluctuate. Our paper aims to answer these questions by proposing a new consumption based asset pricing model.

Central banks around the world rely on reduced form Gaussian affine term structure models (ATSM) to produce estimates of term premia for policy discussion, because they are econometrically flexible and economically interpretable. This class of models generate variability in term premia through a time-varying risk price that is a function of the conditional mean of yields; see, e.g. Duffee (2002), Wright (2011) and Bauer et al. (2012). Conversely, structural consumption-based asset pricing models studying bond risk premia often use recursive preferences, Bansal and Shaliastovich (2013) for example. In these models, time-variation in term premia are driven only by stochastic volatility of consumption growth and inflation, meaning that the levels of these variables and yields themselves play no role in generating risk premia. This conclusion is at odds with the empirical evidence from reduced form ATSMs because it shuts down the channel that makes reduced-form ATSMs empirically successful.

We reconcile the two literatures by introducing both time-varying risk prices and quantities of risk into a consumption-based model, where the former are functions of the conditional mean of the macroeconomic factors. We introduce a time-varying risk price using a new formulation of a stochastic rate of time preference. Unlike Albuquerque et al. (2016) and Schorfheide et al. (2014), our preference shock only depends on macroeconomic variables, current and past consumption and inflation, and not on any additional latent factors. Our specification implies that inflation can be non-neutral. Similar to Piazzesi and Schnei-

der (2007) and Bansal and Shaliastovich (2013), we find that this is a key ingredient for generating an upward sloping yield curve and a realistic pattern of term premia.

The novelty of our specification for the preference shock is that bond prices remain analytical and retain an affine structure. The tractability gained from the affine structure allows us to empirically disentangle the roles that the preference shock, recursive preferences, and stochastic volatility have on term premia. We introduce time-varying quantities of risk through stochastic volatility, similar to the long run risk literature. The difference is that the volatility process in our paper is guaranteed to remain positive, unlike most of the literature.

The empirical examination of the asset pricing implications requires solving for the stochastic discount factor.<sup>1</sup> However, such a solution does not always exist for regions of the parameter space that researchers have traditionally found plausible. This is an issue for all models in the literature that use recursive preferences with a few exceptions, and not specific to our model. We provide conditions on the model's parameters guaranteeing the existence of a solution. These conditions partition the parameter space and make it cumbersome for econometricians implementing either an optimization-based frequentist estimator or Bayesian Markov chain Monte Carlo algorithm.

We evaluate the empirical performance of our model by first quantifying the information contained in observed inflation and consumption data about macroeconomic state variables. We use the particle Gibbs sampler to efficiently estimate the state variables. We then ask how well these macroeconomic state variables do in terms of matching the bond yield data by least squares. We show that empirically our model can adequately capture the time-variation of term premia. It fits other key moments of Treasury bonds as well: it has an upward slope for the yield curve; and it also mimics the time series dynamics of the average yield and slope of the yield curve well.

Next, we turn to the question that motivates the paper: is a time-varying risk price that

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<sup>1</sup>Specifically, we implement a solution method developed by Bansal and Yaron (2004) that is widely used in the macroeconomics and finance literatures; see, e.g. Bollerslev et al. (2009), Bansal et al. (2012) and Schorfheide et al. (2014). Rudebusch and Swanson (2012) and Caldara et al. (2012) describe other solution methods.

is a function of the conditional mean of macro variables or quantity of risk the key driver for the time variation in term premia? We answer this question by shutting down one channel at a time. First, we shut down the price of risk channel. This model is similar to a long run risk model with stochastic volatility studied in the literature. We find that although such a model produces time variation in term premia, the implied term premia are implausible: they are economically insignificant, and have the wrong sign.

On the other hand, a model with the preference shock but not stochastic volatility mimics the time variation of term premia produced by the reduced form Gaussian ATSM and our benchmark model. Overall, our empirical evidence attributes the time variation in the term premia primarily to a time-varying risk price through the preference shock.

We further examine whether it is inflation or consumption growth that drives this risk price, and we find the crucial component is the price of the expected inflation risk that comoves with the expected inflation itself. This is consistent with inflation non-neutrality as argued by Piazzesi and Schneider (2007) and Bansal and Shaliastovich (2013) in a structural framework. The important contribution inflation makes to bond price dynamics are also highlighted in the ATSM literature, see Ang and Piazzesi (2003) and Rudebusch and Wu (2008).

Introducing the preference shock also has important implications for the unconditional slope of the yield curve, which has been the main focus for the majority of the literature. Once the macro factors are pinned down by the observed macro data themselves, we show the long run risk model with stochastic volatility implies a counterfactual downward sloping nominal yield curve. Adding the preference shock reverts the result completely, and implies an unconditional slope just like what we see in the data.

This paper continues as follows. We introduce the new model with the preference shock and recursive preferences in [Section 2](#), and discusses its properties in [Section 3](#). In [Section 4](#), we discuss the conditions for a solution to exist. [Section 5](#) describes empirical strategy for estimation and the consequent estimates. [Section 7](#) examines the model's implications for

bonds. The paper concludes in [Section 8](#). Details on the model, solution mechanism, and estimation method are available in an online appendix.

## 2 Model

In this section, we build a model that encompasses the two competing channels that drive time variation in term premia: time-varying risk prices and quantities of risk. Unlike a standard model with recursive preferences, our model generates a time-varying risk price that is a function of macroeconomic variables. Moreover, our model is tractable because it is affine. The derivations for the equations in this section can be found in the online appendix.

### 2.1 Agent's problem

We consider an endowment economy, where the representative agent optimizes over his lifetime utility

$$V_t = \max_{C_t} \left[ (1 - \beta) \Upsilon_t C_t^{1-\eta} + \beta \{E_t [V_{t+1}^{1-\gamma}]\}^{\frac{1-\eta}{1-\gamma}} \right]^{\frac{1}{1-\eta}}, \quad (2.1)$$

with respect to consumption  $C_t$ . Like Albuquerque et al. (2016) and Schorfheide et al. (2014),  $\Upsilon_t$  introduces variation in the rate of time preference, which is captured by  $\beta$ , and we will refer to  $\Upsilon_t$  as the stochastic rate of time preference. The parameter  $\gamma$  measures risk aversion, and  $\psi = \frac{1}{\eta}$  is the elasticity of intertemporal substitution when there is no uncertainty. Agents maximize utility (2.1) subject to the budget constraint

$$W_{t+1} = (W_t - C_t) R_{c,t+1}, \quad (2.2)$$

where  $W_t$  is wealth and  $R_{c,t+1}$  is the gross return on the consumption asset between  $t$  and  $t + 1$ .

The first order condition of the agent's problem implies that the log stochastic discount

factor (SDF) is

$$m_{t+1} = \vartheta \ln(\beta) + \vartheta \Delta v_{t+1} - \eta \vartheta \Delta c_{t+1} + (\vartheta - 1) r_{c,t+1}, \quad (2.3)$$

where  $\vartheta \equiv \frac{1-\gamma}{1-\eta}$ ,  $\Delta c_{t+1} = \ln(C_{t+1}) - \ln(C_t)$  is consumption growth, and  $r_{c,t+1} = \ln(R_{c,t+1})$  is the continuously compounded return. These terms are standard in models with recursive preferences. The preference shock  $\Delta v_{t+1} = \ln \Upsilon_{t+1} - \ln \Upsilon_t$  is the key term. Our formulation enables variation in the pricing kernel to capture the time-varying risk premium in bond prices through a time-varying risk price. Nominal assets are priced using the nominal pricing kernel

$$m_{t+1}^{\$} = m_{t+1} - \pi_{t+1}, \quad (2.4)$$

where inflation is  $\pi_{t+1} = \ln(\Pi_{t+1}) - \ln(\Pi_t)$  and  $\Pi_t$  is the nominal price level.

## 2.2 Dynamics

In this section, we describe the dynamics of inflation and consumption, and then map it into a companion form. We model consumption growth  $\Delta c_t$  as the sum of a long-run risk component  $\bar{c}_t$  and an idiosyncratic measurement error  $\varepsilon_{c_1,t}$  as in Bansal and Yaron (2004). We use a similar model for inflation  $\pi_t$  with expected inflation  $\bar{\pi}_t$  and an idiosyncratic shock  $\varepsilon_{\pi_1,t}$ . The model is

$$\pi_{t+1} = \bar{\pi}_t + \sqrt{h_{t,\pi}} \varepsilon_{\pi_1,t+1} \quad (2.5)$$

$$\Delta c_{t+1} = \bar{c}_t + \sqrt{h_{t,c}} \varepsilon_{c_1,t+1} \quad (2.6)$$

$$\bar{\pi}_{t+1} = \mu_{\pi} + \phi_{\pi} \bar{\pi}_t + \phi_{\pi,c} \bar{c}_t + \sigma_{\pi} \sqrt{h_{t,\pi}} \varepsilon_{\pi_2,t+1} \quad (2.7)$$

$$\bar{c}_{t+1} = \mu_c + \phi_{c,\pi} \bar{\pi}_t + \phi_c \bar{c}_t + \sigma_{c,\pi} \sqrt{h_{t,\pi}} \varepsilon_{\pi_2,t+1} + \sigma_c \sqrt{h_{t,c}} \varepsilon_{c_2,t+1} \quad (2.8)$$

where the shocks  $\{\varepsilon_{\pi_1,t}, \varepsilon_{c_1,t}, \varepsilon_{\pi_2,t}, \varepsilon_{c_2,t}\}$  are independent and standard normal. In this model, shocks to expected inflation have a contemporaneous impact on expected consumption growth, and all shocks have stochastic volatility. The conditional variances  $h_{t,\pi}$  and  $h_{t,c}$  follow a non-central gamma (NCG) process as in Creal and Wu (2015b). This process guarantees the non-negativity of volatility. This contrasts with the Gaussian process that is prevalent in most of the literature, for example, see Bansal and Yaron (2004) and Bansal and Shaliastovich (2013).

We write the model (2.5) - (2.8) in companion form by defining consumption growth and inflation as a linear function of a state vector  $g_t$  which follows a heteroskedastic vector autoregression

$$\Delta c_t = Z'_c g_t, \tag{2.9}$$

$$\pi_t = Z'_\pi g_t, \tag{2.10}$$

$$g_{t+1} = \mu_g + \Phi_g g_t + \Phi_{gh} h_t + \Sigma_{gh} \varepsilon_{h,t+1} + \Sigma_{g,t} \varepsilon_{g,t+1}, \quad \varepsilon_{g,t+1} \sim N(0, I) \tag{2.11}$$

$$\Sigma_{g,t} \Sigma'_{g,t} = \Sigma_{0,g} \Sigma'_{0,g} + \sum_{i=1}^H \Sigma_{i,g} \Sigma'_{i,g} h_{it},$$

$$h_{t+1} \sim \text{NCG}(\nu_h, \Phi_h, \Sigma_h), \tag{2.12}$$

$$\varepsilon_{h,t+1} = h_{t+1} - E_t(h_{t+1}|h_t).$$

where  $Z_c$  and  $Z_\pi$  are  $G \times 1$  selection vectors. Mapping the model (2.5) - (2.8) to the companion form, the state vector is  $g_t = (\pi_t, \Delta c_t, \bar{\pi}_t, \bar{c}_t)'$  and the  $H \times 1$  vector of conditional variances is  $h_t = (h_{t,\pi}, h_{t,c})'$ . See the online appendix for more details. The general process (2.11) - (2.12) nests popular models in the literature for consumption growth including the vector autoregressive moving average model of Wachter (2006).

The stochastic volatility model for  $h_t$  is an affine process that is the exact discrete time equivalent of a multivariate Cox et al. (1985) process. The conditional mean is  $E_t[h_{t+1}|h_t] = \Sigma_h \nu_h + \Phi_h h_t$  meaning that  $\Phi_h$  controls the autocovariance of  $h_{t+1}$  and  $\Sigma_h \nu_h$  is the drift.  $\Sigma_h$  is a matrix of scale parameters and  $\nu_h$  are a vector of shape parameters. The vector  $\varepsilon_{h,t+1}$

are mean zero, heteroskedastic shocks to volatility and  $\Sigma_{gh}$  measures the covariance between Gaussian and non-Gaussian shocks, i.e. the volatility feedback effect.

## 2.3 Preference shock

Empirical evidence from the term structure literature using affine models shows that risk premia are driven by the levels of the state variables through a time-varying risk price. We build this channel in our model through the preference shock. In this paper, we use the term “preference shock” differently than in Albuquerque et al. (2016) and Schorfheide et al. (2014). In their models, the preference shock is an autocorrelated latent factor impacting the rate of time preference. In our model, the rate of time preference is also stochastic but it depends on the levels of macroeconomic variables and not a latent factor. The dependence on inflation is motivated by the real effect of inflation as argued by Piazzesi and Schneider (2007) and Bansal and Shaliastovich (2013). We will empirically demonstrate its importance in our context for generating realistic term premia dynamics in [Subsection 7.1](#).

The preference shock is specified as follows:

$$\Delta v_{t+1} = \Lambda_1(g_t) + \Lambda_2(g_t)' \varepsilon_{g,t+1}, \quad (2.13)$$

which depends on the levels of past inflation and consumption growth as well as their shocks in the current period. We parameterize the functions in [\(2.13\)](#) as:

$$\Lambda_2(g_t) = -\eta \Sigma_{g,t}^{-1} (\lambda_0 + \lambda_g g_t), \quad (2.14)$$

where  $\Lambda_1(g_t) = -\frac{\vartheta \eta^2}{2} (\lambda_0 + \lambda_g g_t)' (\Sigma_{g,t} \Sigma_{g,t}')^{-1} (\lambda_0 + \lambda_g g_t)$  cancels out the Jensen’s inequality term. The novelty in our choice of the functional form is two fold. First, it allows the model to stay within the affine family and have analytical bond prices. Second, using macro variables allows us to understand the driving force in macroeconomic fundamentals.

This specification can be readily extended to a more complex model where the preference



shock  $\Delta v_{t+1}$  also depends on the volatilities and their shocks, see the earlier working paper version, Creal and Wu (2015a). This extension could potentially introduce more complex channels to explain movements in asset prices. Instead, we intentionally keep the model simple and examine how well the current channels can explain yields. Compared to models with recursive preferences, our model is more flexible because it introduces an additional channel (the preference shock) through which a representative agent perceives risk. Nevertheless, our model is still significantly restricted compared to a standard no-arbitrage affine model because it has no latent yield factors.

## 2.4 Solving $r_{c,t+1}$

The SDF in (2.3) is a function of the return on the consumption asset  $r_{c,t+1}$ , which is generally regarded as unobserved in the data. We solve for it using the log-linearization technique of Campbell and Shiller (1989), applied by, for example, Bansal and Yaron (2004) and Bansal et al. (2012).<sup>2</sup> We express the return as a function of the price to consumption ratio

$$\begin{aligned} r_{c,t+1} &\equiv \ln \left( \frac{P_{t+1} + C_{t+1}}{P_t} \right) = \Delta c_{t+1} - pc_t + \ln(1 + \exp(pc_{t+1})) \\ &\approx \kappa_0 + \kappa_1 pc_{t+1} - pc_t + \Delta c_{t+1}, \end{aligned} \tag{2.15}$$

where  $P_{t+1}$  is the price of consumption goods,  $pc_t = \ln \left( \frac{P_t}{C_t} \right)$  is the log price to consumption ratio. The parameters  $\kappa_0$  and  $\kappa_1$  are log-linearization constants that depend on the average price to consumption ratio  $\bar{pc} = E(pc_t)$ .

As the real pricing kernel in (2.3) must also price the consumption good

$$1 = E_t [\exp(m_{t+1} + r_{c,t+1})] \tag{2.16}$$

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<sup>2</sup>The solution method used by Campbell et al. (2014) for their ICAPM model is similar, only they substitute out consumption instead of the return on the consumption asset.

we can guess and verify a solution for

$$pc_t = D_0 + D'_g g_t + D'_h h_t. \quad (2.17)$$

Together, (2.15) and (2.17) express  $r_{c,t+1}$  and hence the pricing kernel  $m_{t+1}$  as functions of the underlying state variables  $g_t$  and  $h_t$ . This is a fixed point problem:  $pc_t$  depends on  $\kappa_0, \kappa_1$  through  $D_0, D_g, D_h$ , which in turn depend on  $\bar{p}c$ . This fixed point problem makes the model more complex as the parameter space needs to be restricted to the region where a solution exists. We discuss the details of this fixed point problem in the online appendix and the existence of a solution in [Section 4](#).

### 3 Model properties

This section examines the different components contributing to time variation in bond risk premia. Specifically, we examine how the preference shock specification in the previous section translates into a time-varying risk price, and hence time-varying risk premium.

#### 3.1 Sources of risk premia

Using the solution method described in [Subsection 2.4](#), the nominal log-SDF in deviation from the mean form becomes

$$m_{t+1}^{\$} - E_t(m_{t+1}^{\$}) = -\lambda_{g,t}^{\$,'} \Sigma_{g,t} \varepsilon_{g,t+1} - \lambda_h^{\$,'} \Sigma_{h,t} \tilde{\varepsilon}_{h,t+1} \quad (3.1)$$

where the vector of shocks to volatility  $\tilde{\varepsilon}_{h,t+1} = \Sigma_{h,t}^{-1} \varepsilon_{h,t+1}$  have been standardized to have unit variance. Both the Gaussian and non-Gaussian shocks are heteroskedastic and have a time-varying quantity of risk.

Due to the stochastic rate of time preference, shocks to the SDF have time-varying risk

price  $\lambda_{g,t}^{\$}$  given by

$$\begin{aligned}
\lambda_{g,t}^{\$} &= \gamma Z_c + Z_\pi && \leftarrow \text{power utility} \\
&+ \kappa_1 \frac{\gamma - \eta}{1 - \eta} D_g && \leftarrow \text{recursive preferences} \\
&+ \vartheta \eta (\Sigma_{g,t} \Sigma'_{g,t})^{-1} (\lambda_0 + \lambda_g g_t). && \leftarrow \text{preference shock} \quad (3.2)
\end{aligned}$$

The first two terms are inherited from power utility, the second line comes from recursive preferences, and the third is due to the preference shock.

The key term in (3.2) is  $\lambda_g g_t$ . The model has a time-varying risk price and produces time-varying term premia that are functions of  $g_t$  only if  $\lambda_g$  is non-zero. This channel still exists even when we shut off the stochastic volatility in the dynamics (2.11) - (2.12). Conversely, if there were no preference shock, i.e.  $\lambda_0 = 0, \lambda_g = 0$ , the risk price  $\lambda_{g,t}^{\$}$  is constant as in the literature on consumption-based models with recursive preferences. A time-varying risk price that co-moves with the levels of macroeconomic variables and yields is a feature of benchmark Gaussian affine models that makes it successful empirically, and it is a feature that is absent in standard models with recursive preferences.

If there were no preference shock, i.e. if  $\lambda_0 = 0$  and  $\lambda_g = 0 \implies \Upsilon_t = 1$  in (2.1), then the price of short run consumption growth risk is equal to the risk aversion coefficient  $\gamma$ , the price of short run inflation risk is 1 for the nominal pricing kernel, and 0 for the real pricing kernel. These are consistent with the literature.

Next, we decompose the risk price for the non-Gaussian shocks as

$$\begin{aligned}
\lambda_h^{\$} &= \Sigma'_{gh} (\gamma Z_c + Z_\pi) && \leftarrow \text{power utility} \\
&+ \kappa_1 \frac{(\gamma - \eta)}{(1 - \eta)} (\Sigma'_{gh} D_g + D_h) && \leftarrow \text{recursive preference}
\end{aligned}$$

These terms have similar features and functional forms as those in (3.2). Power utility only has an impact on the risk price of volatility if there is a volatility feedback effect with

$\Sigma_{gh} \neq 0$ , while recursive preferences generate a risk price even when  $\Sigma_{gh} = 0$ . To keep our model simple and tractable, as in the standard models with recursive preferences, our model does not have a time-varying risk price for volatility. Like the standard models, we also have time-varying quantities of risk through stochastic volatility.

### 3.2 Bond prices and term premium

The price of a zero-coupon nominal bond with maturity  $n$  at time  $t$  is the expected price of the same asset at time  $t + 1$  discounted by the stochastic discount factor

$$P_t^{\$, (n)} = \mathbb{E}_t \left[ \exp(m_{t+1}^{\$}) P_{t+1}^{\$, (n-1)} \right]. \quad (3.3)$$

Following Creal and Wu (2015b), nominal yields are an affine function of both the Gaussian state vector and volatility

$$y_t^{\$, (n)} = a_n^{\$} + b_{n,g}^{\$, \prime} g_t + b_{n,h}^{\$, \prime} h_t, \quad (3.4)$$

where the key coefficient is  $b_{n,g}^{\$} = -\frac{1}{n} \bar{b}_{n,g}^{\$}$ , and  $\bar{b}_{n,g}^{\$}$  follows the recursion

$$\bar{b}_{n,g}^{\$} = (\Phi_g - \eta \vartheta \lambda_g)' \bar{b}_{n-1,g}^{\$} + \bar{b}_{1,g}^{\$}. \quad (3.5)$$

The bond-loadings are similar to those found in Gaussian affine models, see the online appendix.

One key insight from the term structure literature is the difference between the autocorrelations under the risk neutral and physical measures; given by  $\Phi_g^{\mathbb{Q}^{\$}} \equiv \Phi_g - \eta \vartheta \lambda_g \neq \Phi_g$ . The basic intuition is that the autoregressive coefficient driving the physical dynamics  $\Phi_g$  is separated from the parameters that determine the cross section (slope) of the yield curve  $\Phi_g^{\mathbb{Q}^{\$}}$ . It is this feature that enables Gaussian affine models to fit the data and generate enough variation in term premia. We demonstrate how to generate this key channel with a

stochastic rate of time preference in a structural model.

Another contribution of our paper is to map a consumption-based model with the preference shock and recursive preferences into an affine framework. The benefit of this is two fold. First, the analytical structure makes it transparent how the alternative mechanisms that can drive risk premia work inside the model. Secondly, it makes the model more tractable empirically.

The nominal term premium is defined as the difference between the model implied yield  $y_t^{\$, (n)}$  and the average of expected future short rates over the same period

$$tp_t^{\$, (n)} = y_t^{\$, (n)} - \frac{1}{n} E_t (r_t^{\$} + r_{t+1}^{\$} + \dots + r_{t+n-1}^{\$}). \quad (3.6)$$

The term premium has a simple portfolio interpretation. An investor can buy an  $n$ -period bond and hold it until maturity or he can purchase a sequence of 1 period bonds, repeatedly rolling them over for  $n$  periods. The term premium measures the additional compensation a risk averse agent needs to choose one option over another. Empirical evidence in the term structure literature shows the term premium is time-varying, which implies  $\lambda_g \neq 0$  in (2.14).

## 4 Feasible parameter regions

Empirical examination of the asset pricing implications of recursive preferences requires solving for the SDF. We base our analysis on the approximation method of Campbell and Shiller (1989), used by Bansal et al. (2012) and Schorfheide et al. (2014), among many others. With this approach, we need to solve for the return on the consumption asset  $r_{c,t+1}$  as a function of the underlying state of the economy as described in [Subsection 2.4](#). Whether such a solution exists amounts to a fixed point problem. In this section, we characterize the

conditions that lead to a valid solution for the Euler equation and asset prices.<sup>3</sup>

We partition the vector of all parameters of the model  $\theta = (\beta, \psi, \gamma, \theta^{\mathbb{P}}, \theta^{\lambda})$  into the preference parameters  $(\beta, \psi, \gamma)$ , the parameters governing the physical dynamics  $\theta^{\mathbb{P}}$ , and the parameters controlling the preference shock  $\theta^{\lambda}$ . We condition our analysis on  $(\theta^{\mathbb{P}}, \theta^{\lambda})$  and characterize the restrictions on the parameter space for the more intuitive parameters  $(\beta, \psi, \gamma)$ .

## 4.1 General case

Laid out briefly in [Subsection 2.4](#), the fixed point problem can be rephrased more explicitly as follows. Define the function  $f(\bar{p}c, \theta)$  as

$$f(\bar{p}c; \theta) = D_0(\bar{p}c, \theta) + D_g(\bar{p}c, \theta)' \bar{g} + D_h(\bar{p}c, \theta)' \bar{h}. \quad (4.1)$$

For a given value of  $\theta$ , a solution to the fixed point problem is obtained when  $f(\bar{p}c; \theta) = \bar{p}c$ . Such a solution does not always exist. Instead, the parameters must lie in a restricted space that ensures a solution.

Before we discuss the solution for the fixed point problem to exist, the parameters need to satisfy some conditions that are specific to models with stochastic volatility.<sup>4</sup>

**Assumption 1** *The parameters  $\theta \in \Theta^r$  must satisfy that for any real  $\bar{p}c$ ,*

1. *the loadings  $D_h(\bar{p}c, \theta)$  are real,*
2. *the expectation in (2.16) exists for  $D_h(\bar{p}c, \theta)$ .*

The first part of the assumption is used to guarantee that  $f(\bar{p}c, \theta)$  is real. It amounts to a real solution for a system of  $H$  quadratic equations in  $H$  unknowns, i.e. their respective

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<sup>3</sup>Hansen and Scheinkman (2012) also discuss conditions that guarantee a solution to the representative agent's problem under recursive preferences.

<sup>4</sup>Models with rare consumption disasters with time-varying jump intensities following a Cox et al. (1985) process will require similar conditions.

discriminant must be positive.<sup>5</sup> Second, the guess and verify technique used to solve the coefficients in (2.17) requires the expectation in (2.16) to exist. This expectation does not always exist when stochastic volatility follows a multivariate Cox et al. (1985) process, and the second part of Assumption 1 guarantees the existence of the integral. These conditions are discussed in more detail in the online appendix.

Given these conditions, the following proposition provides a general condition that guarantees a solution to the representative agent's problem.

**Proposition 1** *Given Assumption 1, there is a value  $\bar{\beta}(\psi, \gamma, \theta^P, \theta^\lambda)$  such that if  $\beta < \bar{\beta}$ , then there exists a real solution for the fixed point problem.*

**Proof:** See the online appendix.

We use the proposition to characterize the joint restrictions that exist among all the parameters. Given the dynamics of the economy in  $\theta^P$  and the parameters determining the preference shock in  $\theta^\lambda$ , agents' risk appetite  $\gamma$ , and the intertemporal elasticity of substitution  $\psi$ , the representative agent needs to be sufficiently impatient (small  $\beta$ ) in order for a solution to exist. The nature of the fixed point problem requires that all three conditions be jointly satisfied. The proposition above provides sufficient conditions for a solution to exist. One can check the uniqueness of the solution numerically using the method proposed in Doh (2013). In a subsequent work, Borovicka and Stachurski (2017) developed conditions for the existence and uniqueness of recursive utility for a slightly different class of dynamics of the state variables.

## 4.2 Special cases

In this section, we provide more intuition by discussing a special case where the dynamics are Gaussian by imposing  $h_t = 0$  in (2.11) - (2.12). In this case, we are no longer constrained by

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<sup>5</sup>This condition is similar to an existence condition discussed by Campbell et al. (2014) in their ICAPM model. They do not provide a condition guaranteeing a solution to the fixed point problem.

Assumption 1. We can provide stronger conditions that apply to any  $\beta \leq 1$ , i.e., it reduces to relationships between  $\gamma$  and  $\psi$ . The following corollary also characterizes the upper bound  $\bar{\beta}$  as a monotonic function in  $\gamma$ .

**Corollary 1** 1. If  $Z_1^{\infty} \mu_g^* \leq 0$  and  $\beta \leq 1$ , then  $\frac{1-\gamma}{1-\psi} > 0$  guarantees the existence of a solution.

2. If  $\beta \leq 1$ , then there is a value  $\bar{\gamma}(\theta^{\mathbb{P}}, \theta^{\lambda})$  such that  $\frac{\bar{\gamma}-\gamma}{1-\psi} > 0$  guarantees a solution.

3. For any  $\psi$ ,  $\bar{\beta}$  is monotonic in  $\gamma$ : for  $\psi > 1$ , then  $\frac{d\bar{\beta}}{d\gamma} > 0$ ; for  $\psi < 1$ , then  $\frac{d\bar{\beta}}{d\gamma} < 0$ .

Under the condition specified in part 1 of Corollary 1, a solution exists if  $(\gamma > 1, \psi > 1)$  or  $(\gamma < 1, \psi < 1)$ . This divides the parameter space for  $(\gamma, \psi)$  into four quadrants, and only two of these four have a solution. Part 2 of Corollary 1 says that  $(\gamma > \bar{\gamma}, \psi > 1)$  or  $(\gamma < \bar{\gamma}, \psi < 1)$  guarantees a solution, regardless of how patient the agent is. Again, two out of the four quadrants have a solution, similar to part 1. The intuition behind the two parts of the Corollary is also similar. Although the cutoff for  $\psi$  is always 1, the difference between them is the boundary on  $\gamma$  in part 2 depends on the parameters  $\theta^{\mathbb{P}}$  and  $\theta^{\lambda}$ .

The separation of the parameter space into quadrants makes estimation more challenging. For example, if the optimum is within the upper-right region and we start from the lower left region, a numerical optimization algorithm or a Bayesian MCMC algorithm, can have a hard time getting through the tiny bottleneck and reaching the correct part of the parameter space. In practice, we observe these algorithms hitting the regions where no solution exists and often stopping. Estimation gets more complicated when the structural parameters interact with the remaining parameters of the model as the boundaries can shift creating strong dependencies among the model's parameters.

Corollary 1 part 3 states the relationship between the upper bound for  $\beta$  and  $\gamma$ . If  $\psi < 1$ , then an agent cannot have a high risk aversion and be patient at the same time. The more risk averse he is, the less patient he needs to be, and vice versa. If  $\psi > 1$ , the opposite is true. Numerical illustrations can be found in the online appendix.



## 5 Estimation

We estimate the model using a procedure similar to Piazzesi and Schneider (2007), where we first estimate a stochastic process for consumption growth and inflation and then use these macroeconomic state variables to fit the yield curve. The differences are how we implement the steps. First, we use Bayesian procedures to estimate the state variables because of the presence of stochastic volatility. Next, instead of calibrating the structural parameters as in Piazzesi and Schneider (2007), we use non-linear least squares to estimate them by minimizing a quadratic form in the average pricing errors. This approach ensures that the macroeconomic variables (expected inflation, expected consumption growth, and their stochastic volatilities) maintain their intended economic interpretation because it only uses macroeconomic data to estimate them.

### 5.1 Data

The data we use are standard in the literature. Our measure of monthly real per capita consumption growth is constructed from nominal non-durables and services data downloaded from the NIPA tables at the U.S. Bureau of Economic Analysis. We deflate each of these series by their respective price indices, add them together, and divide by the civilian population. The population series and monthly U.S. CPI inflation are downloaded from the Federal Reserve Bank of St. Louis. Yields are the Fama and Bliss (1987) zero coupon bond data available from the Center for Research in Securities Prices (CRSP) with maturities of (3, 12, 24, 36, 48, 60) months. The data spans from February 1959 through June 2014 for a total of  $T = 665$  observations.

### 5.2 Parameter restrictions

In the model with stochastic volatility,  $\Sigma_{0,g}$  is set to zero for identification. We also impose  $\Phi_h$ ,  $\Sigma_h$  to be diagonal, and  $\Phi_{gh} = 0$  and  $\Sigma_{gh} = 0$  to reduce the number of parameters. In

the Gaussian model without stochastic volatility, the parameters of  $\Sigma_{0,g}$  are estimated.

If there is no preference shock, there are three structural parameters  $(\beta, \psi, \gamma)$  to fit the cross section of yields. In models with the preference shock, we only introduce four new parameters into the matrix  $\lambda_g$  (the elements related to expected consumption, expected inflation, and their cross terms) while we set  $\lambda_0 = 0$ .

Even with the new parameters in  $\lambda_g$ , the new model still imposes more restrictions than a Gaussian no arbitrage affine model. For example, if we compare the two classes of models assuming the same dynamics for consumption growth and inflation as in this setting, there are 34 free parameters in a Gaussian affine model while there are 26 parameters in our model. No-arbitrage term structure models impose considerably less structure under the  $\mathbb{Q}$  dynamics than that implied by a structural model. Equivalently, the bond loadings contain more free parameters to fit asset prices.

### 5.3 Estimation of consumption growth and inflation dynamics

We estimate the model of consumption growth and inflation (2.5) - (2.8) by Bayesian methods. We place a prior distribution over the parameters of the model and estimate them along with the state variables  $g_t$  and  $h_t$  by drawing from the posterior distribution using Markov chain Monte Carlo and a particle Gibbs sampler. A detailed description of how the model is estimated as well as empirical results for the model can be found in the online appendix.

### 5.4 Cross sectional regression

Stacking yields (3.4) for  $N$  different maturities  $n_1, n_2, \dots, n_N$  and adding a vector of pricing errors  $e_t$ , the observation equations for yields are

$$y_t^{\$} = A^{\$} + B_g^{\$}g_t + B_h^{\$}h_t + e_t, \quad e_t \sim \text{i.i.d. } (0, \omega^2 I) \quad (5.1)$$

where  $y_t^{\$} = (y_t^{\$, (n_1)}, y_t^{\$, (n_2)}, \dots, y_t^{\$, (n_N)})$ ,  $A^{\$} = (a_{n_1}^{\$}, \dots, a_{n_N}^{\$})'$ ,  $B_g^{\$} = (b_{g, n_1}^{\$, \prime}, \dots, b_{g, n_N}^{\$, \prime})'$ , and  $B_h^{\$} = (b_{h, n_1}^{\$, \prime}, \dots, b_{h, n_N}^{\$, \prime})'$ .

Given the estimated parameters  $\hat{\theta}^{\text{P}}$  and the estimates of the state variables  $\hat{g}_t$  and  $\hat{h}_t$ , we estimate the structural parameters  $\theta^{\text{Q}} = (\beta, \gamma, \psi, \theta^{\lambda})$  through non-linear least squares. We compute the pricing error for the vector of yields

$$e_t = y_t^{\$} - A^{\$}(\theta^{\text{Q}}, \hat{\theta}^{\text{P}}) - B_g^{\$}(\theta^{\text{Q}}, \hat{\theta}^{\text{P}})\hat{g}_t - B_h^{\$}(\theta^{\text{Q}}, \hat{\theta}^{\text{P}})\hat{h}_t, \quad (5.2)$$

and estimate  $\theta^{\text{Q}}$  by minimizing the average squared pricing errors

$$\operatorname{argmin}_{\theta^{\text{Q}}} \frac{1}{T} \sum_{t=1}^T e_t' e_t. \quad (5.3)$$

In practice, we optimize over the free parameters in  $\Phi_g^{\text{Q}^{\$}}$  instead of  $\lambda_g$  as we have a better prior knowledge of their scale. We can then solve for  $\lambda_g$  via  $\Phi_g^{\text{Q}^{\$}} = \Phi_g - \eta\vartheta\lambda_g$ .

Table 1 contains parameter estimates from the model. The first column reports the maximum when the state variables  $g_t$  and  $h_t$  are estimated at their posterior means  $\hat{g}_t$  and  $\hat{h}_t$ . The time discount factor  $\beta$  is 1.002, the intertemporal rate of substitution  $\psi$  is 0.8, and the risk aversion parameter  $\gamma$  is 1.7. The risk-neutral autoregressive matrix  $\Phi_g^{\text{Q}^{\$}}$  is much more persistent than its time series counterpart  $\Phi_g$ , with both eigenvalues at 0.985 or greater. This high persistence reflects the smooth and slow decay of the yield curve as a function of maturity.

There exists a long standing debate in the finance literature about the qualitative feature of the structural parameters, for example, whether  $\psi$  is greater or less than one. The long run risk literature (Bansal and Yaron, 2004) argues that values of  $\psi > 1$  should be the case. On the other hand, Campbell (2011) argues the opposite to be consistent with the aggregate evidence. Our estimate of  $\psi$  is less than one, although the standard error is large when the state variables are estimate at their posterior mean. The value of  $\beta$  is estimated close to 1. The risk aversion parameter  $\gamma$  is estimated at 1.7, which is low compared to much of the

Table 1: Parameter estimates

		mean:		10 <sup>th</sup> :		90 <sup>th</sup> :	
$\psi$		0.800 (0.372)		0.742 (0.005)		0.615 (0.001)	
$\beta$		1.002 (0.001)		1.001 (0.0002)		1.003 (0.000)	
$\gamma$		1.704 (0.935)		1.537 (0.014)		0.455 (0.136)	
New	$\Phi_g^{\text{Qs}}$	0.977 (0.017)	-0.023 (0.078)	0.964 (0.005)	-0.192 (0.007)	0.605 (0.000)	0.229 (0.012)
		0.011 (0.009)	1.018 (0.037)	0.020 (0.002)	1.134 (0.004)	0.071 (0.004)	0.951 (0.000)
	$\lambda_g$	0.002 -0.004	0.025 -0.018	0.010 -0.011	0.123 -0.087	-0.268 0.164	0.006 -0.002

Parameters estimates for the benchmark model. First column: the global estimates when macroeconomic factors  $\hat{g}_t$  and  $\hat{h}_t$  are estimated at the posterior mean. Second column: estimates when macroeconomic factors  $\hat{g}_t$  and  $\hat{h}_t$  are estimated at the 10<sup>th</sup> percentile. Third column: estimates when macroeconomic factors  $\hat{g}_t$  and  $\hat{h}_t$  are estimated at the 90<sup>th</sup> percentile. Standard errors are Newey and West (1987).

asset pricing literature.

To capture the uncertainty induced by the two step estimation procedure, we report estimates when the state variables  $g_t$  and  $h_t$  are estimated at their 10<sup>th</sup> and 90<sup>th</sup> percentiles in the last two columns of Table 1. For the 10<sup>th</sup> percentile, the parameters are very similar to the maximum when the state variables are estimated at the posterior mean. At the 90<sup>th</sup> percentile, the estimates of  $\psi$  and  $\beta$  are similar but the estimated value of risk aversion is much lower. Some elements in  $\Phi_g^{\text{Qs}}$  and  $\lambda_g$  also change sign.

## 6 Model-implied yield curve

This section assesses how well the model fits the data in terms of both the unconditional yield curve and its dynamics.

Table 2: Unconditional yield curves

	3	12	24	36	48	60	level	slope
data	4.94	5.33	5.54	5.72	5.88	5.98	5.57	1.04
SV w/ preference shock	5.19	5.26	5.40	5.59	5.83	6.13	5.57	0.94
10 <sup>th</sup>	5.08	5.25	5.47	5.67	5.86	6.07	5.57	0.99
90 <sup>th</sup>	5.11	5.24	5.46	5.68	5.87	6.05	5.57	0.94
Gaussian w/ preference shock	5.08	5.25	5.47	5.69	5.89	6.09	5.58	1.01
SV w/o preference shock	5.65	5.62	5.58	5.54	5.51	5.49	5.57	-0.17

*Average nominal yields in annualized percentage points across time in the data (first row), our benchmark model with both stochastic volatility and preference shock (second to fourth rows), model without stochastic volatility (fifth row), and model without preference shock (last row) for maturities of 3 - 60 months across the columns. The last two columns are the average level of yields across all 6 maturities, and the slope is defined as the difference between the 60 month and 3 month yields.*

## 6.1 Unconditional yield curve

Table 2 contains estimates of the cross section of the yield curve, averaged over time. The top row is the unconditional sample mean of the data. A well established feature of the unconditional nominal yield curve is that it slopes upwards as maturity increases. In our sample, the slope is 1.04%. The second row contains the sample means of the model-implied yields constructed from the posterior mean of the factors. It mimics the first row closely, and implies a slope of 0.94%. The next two rows are the sample means of the model-implied yields constructed using the 10<sup>th</sup> and 90<sup>th</sup> percentiles of the MCMC draws of the macroeconomic factors. The results look similar to the previous two rows. Row 5 is a model that includes the preference shock but shuts down the stochastic volatility channel, and it paints the same picture as our benchmark model.

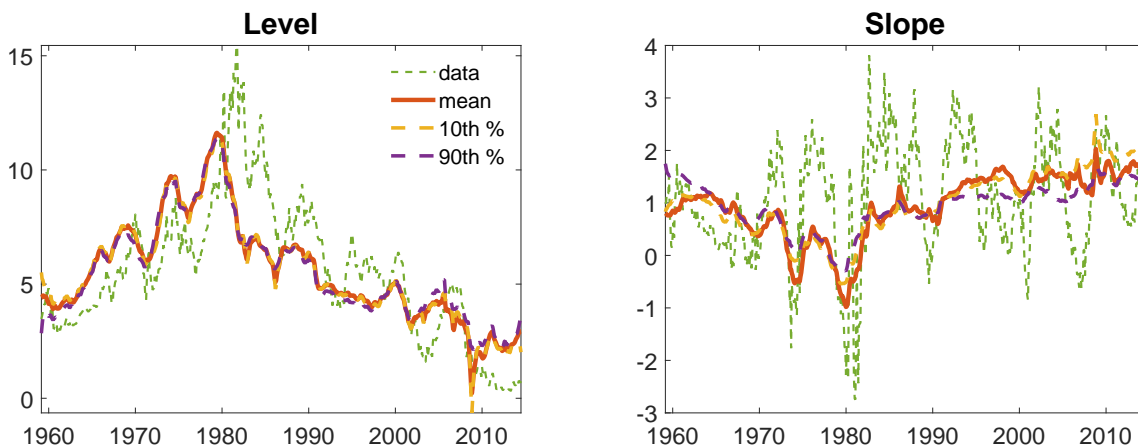
In contrast, if we do not allow preference shocks to play a role by imposing  $\lambda_g = 0$ , then the last row of Table 2 shows a counterfactual downward slope of -0.17%. While this stochastic volatility model seems to be flexible with 2 Gaussian factors and 2 volatility factors, there are only 3 structural parameters  $(\beta, \gamma, \psi)$  to fit the cross section, and all of

them mainly impact bond yields through the intercept term  $a_n^s$  in (3.4). Changes in these parameters allow parallel shifts in the yield curve but are limited in their ability to impact the slope. When  $\lambda_g = 0$ , the autocovariance matrix  $\Phi_g$  determines both the time series dynamics of the factors and the slope of the yield curve at the same time; see the bond loading recursion for  $\bar{b}_{n,g}^s$  in (3.5). In the term structure literature, Duffee (2002) has shown that the separation between the two is important for capturing key features of the data. Using our model, we are able to separate the risk neutral parameter from the times series dynamics  $\Phi_g^{\mathbb{Q}^s} \equiv \Phi_g - \eta\vartheta\lambda_g \neq \Phi_g$  to fit the cross section of the yield curve.

We have demonstrated that our model fits the cross section of the yield curve well when preferences are allowed to be more flexible. The more challenging task is to see if it fits the time series as well. Although term premia have been studied extensively in the term structure literature, they are fundamentally an unobserved object. Consequently, we evaluate our model’s ability to fit the observable moments in the data: level and slope of the yield curve. We will discuss the dynamics of the term premia in [Subsection 7.1](#).

The literature on structural modeling of yields emphasizes the role that the parameter  $\phi_{c,\pi}$  plays in fitting the yield curve; most papers report that  $\phi_{c,\pi}$  should be negative for an upward sloping yield curve. Our model is able to generate an upward sloping yield curve when we use the posterior mean value of  $-0.003$ , which is negative but statistically insignificant. Estimates of the dynamics of the model are available in the online appendix. Our model can generate the same shape for the yield curve even if  $\phi_{c,\pi}$  is positive. We conduct the following exercise: instead of using the mean estimate for  $\phi_{c,\pi}$ , which is negative, we replace it with its 90<sup>th</sup> percentile, which is positive 0.011. Then we re-optimize the objective function, and we find the implied slope is 0.8%. Interestingly, once we introduce the preference shock, our model can fit both cross sectional and time series properties of the yield curve with a positive or negative value of  $\phi_{c,\pi}$ . Moreover, our results are robust if we use multiple volatility factors for each macro variable or a VARMA model for the dynamics of consumption growth and inflation; see online appendix and the earlier working paper version, Creal and Wu (2015a).

Figure 1: Level and slope



*Left: level defined as average of yields across all maturities. Right: slope defined as the 5 year - 3 month yield. Light green dotted line: data; solid red line: mean estimate; yellow dashed line: 10th percentile; purple dashed line: 90th percentile. Y-axis: annualized percentage points.*

## 6.2 Dynamics

In the left panel of [Figure 1](#), we plot the level of the yield curve over time, defined as the average of yields across all maturities in our sample. The red solid line depicts the mean estimate from our model, and the green dashed line is the data. In the right panel, we plot the slope of the yield curve defined as the 5 year yield minus the 3 month yield. Our model implied level and slope trace the data well, considering our model uses only macroeconomic factors rather than latent yield factors. We also plot the model-implied level and slope of the yield curve when the macroeconomic factors are estimated by the 10<sup>th</sup> (yellow dashed lines) and 90<sup>th</sup> (purple dashed lines) percentiles of the MCMC draws. Posterior mean estimates of the level and slope of the yield curve are close to those estimated from the percentiles.

[Table 3](#) shows mean absolute errors for the level and slope of the yield curve. The first row is our model with both preference shock and stochastic volatility, the second row is the Gaussian model with only the preference shock, and the last row is the model with stochastic volatility but without a preference shock. The pricing errors for the level of the

Table 3: Mean absolute errors

		level	slope
Structural	SV w/ preference shock	1.61	0.80
	Gaussian w/ preference shock	1.56	0.85
	SV w/o preference shock	1.59	1.30
Regression	$g_t$	1.60	0.78
	$g_t, h_t$	1.55	0.75

*Mean absolute errors in percentage points. The first row is our model with both stochastic volatility and preference shock. The second row uses the model with only the preference shock. The third row uses the model with only stochastic volatility. The next two rows are regression-based results: the fourth row regresses level or slope on  $g_t$  only; the fifth row regresses level or slope on both  $g_t$  and  $h_t$ .*

yield curve are similar across the three models. The difference is mainly in the slope of the yield curve. While having both stochastic volatility and preference shocks allows our model to fit the slope the best, the Gaussian model with preference shock is a close second. Similar to [Subsection 6.1](#), the pricing error for the slope is much larger when the model has stochastic volatility but no preference shock.

The pricing errors in all the structural models are larger than a typical Gaussian affine model, which is natural for the following reasons. First, structural models are more restrictive than an affine model with fewer free parameters. Second, we restrict our attention to macroeconomic factors. And, our estimated factors are fixed after the first stage of estimation and are functions of only macroeconomic data. We do not allow the factors to adjust to fit the yield curve in the second stage.

The next two rows of [Table 3](#) break down the contributions of the two sources by regressing level or slope on the macroeconomic factors. In these regressions, we are explaining the variation in yields using the macro factors but without imposing any parameter restrictions on the factor loadings that a structural model requires. In row 4, the independent variables are the conditional means of inflation and consumption growth within  $g_t$ . The result in this row is very similar to what we see in the Gaussian model with a preference shock but with no stochastic volatility in row 2. In this case, the restrictions imposed by the structural model



do not introduce significantly higher pricing errors once the preference shock is introduced.

In row 5, we see that adding both conditional mean factors  $g_t$  and variance factors  $h_t$  further improves the pricing errors for both the level and slope when the loadings are unrestricted. Comparing rows 3 and 5 shows that the structural model imposes strong restrictions on the loadings for stochastic volatility. By restricting the loadings on the stochastic volatility factor, the mean absolute errors for the slope increase by 0.55 basis points.

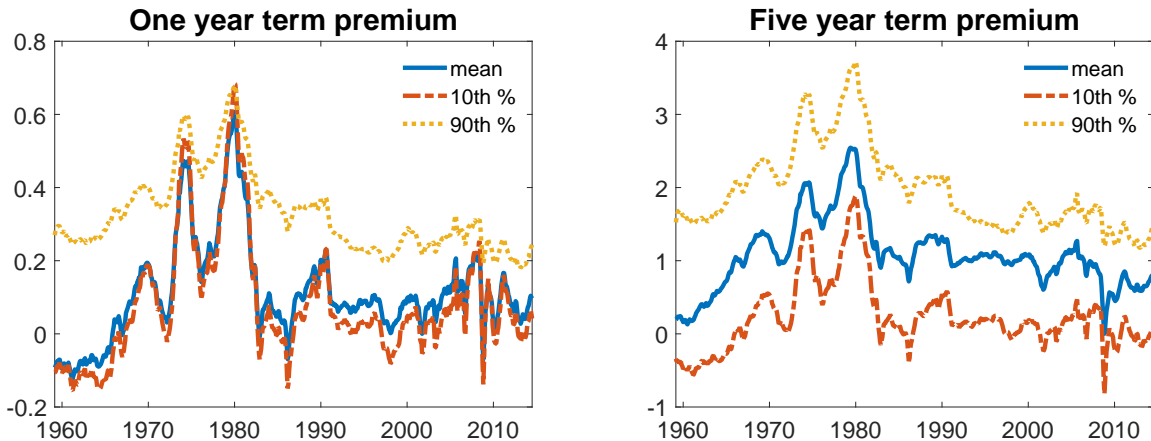
## 7 Bond term premium

Bond term premia are a crucial input for central banks to implement monetary policy and a key object of interest in this paper. In this section, we examine whether the model proposed in [Section 2](#) adequately captures the time-variation of term premia. Then, we decompose this time variation into the alternative channels that contribute to it. Empirically, we find that the key term is the price of expected inflation risk, which loads on expected inflation itself.

### 7.1 Term premium and its sources

We plot the 1 year (left) and 5 year (right) term premia from our main model in [Figure 2](#). The blue solid lines are estimated from the posterior mean of the state variables. The long term (5-year) term premium displays more variation than the medium term (1-year) term premium. The 5-year term premium was low at the beginning of our sample. It increased through the 1960s and 70s, peaked in the early 1980s at about 2.5%, and then it trended down. To capture the uncertainty induced by the two step estimation procedure, we also plot the term premia calculated from the 10<sup>th</sup> and 90<sup>th</sup> percentiles of the MCMC draws. Their dynamics are similar to those of the estimates calculated from the posterior mean. The one exception is the 90<sup>th</sup> percentile for the one year term premium. This is expected from the estimates in [Table 1](#) as the 90<sup>th</sup> percentile produces estimates of the structural parameters

Figure 2: Estimated term premia from our main model



*Estimated 1 and 5 year term premia from our main model with both preference shock and stochastic volatility. Blue solid line: mean estimate; red dashed line: 10th percentile; yellow dotted line: 90th percentile. Y-axis: interest rates measured in annualized percentage points.*

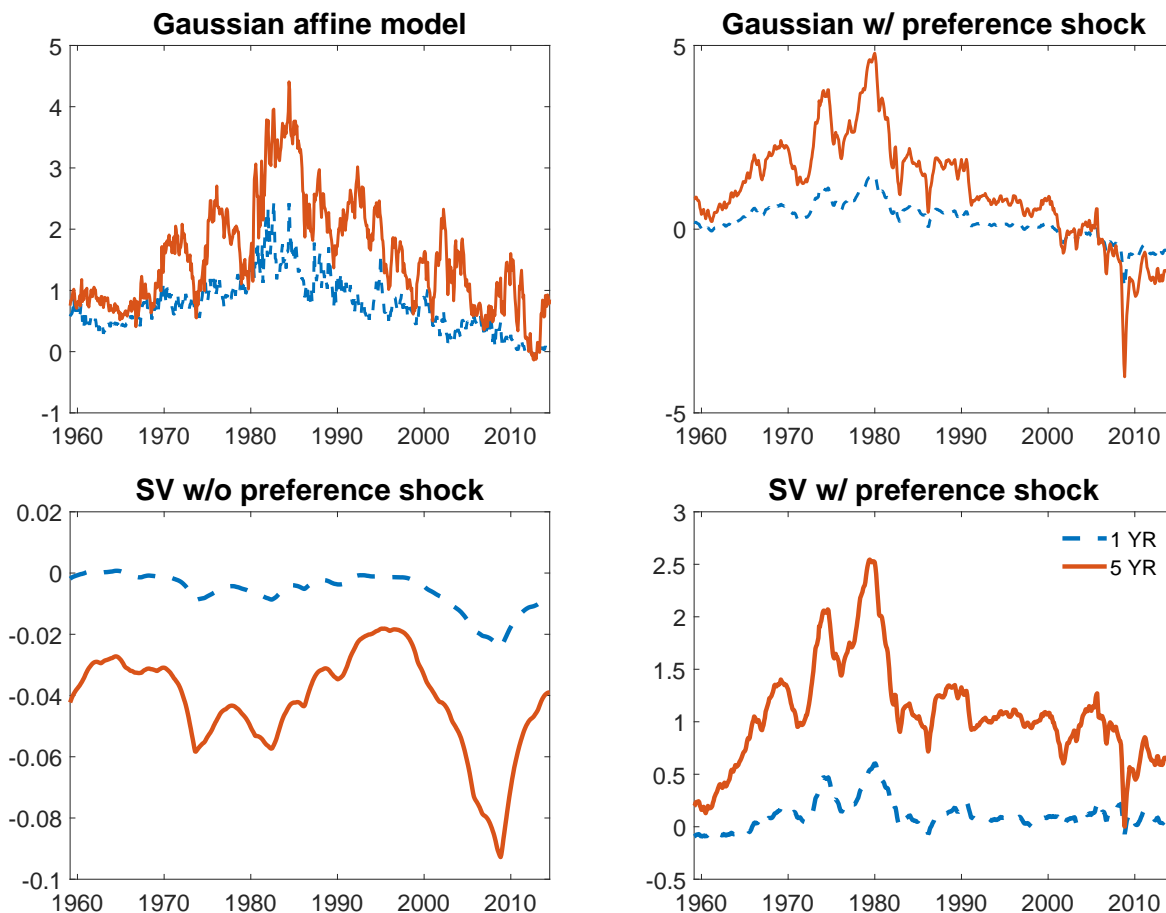
that have different properties than the mean and 10<sup>th</sup> percentile.

For comparison, we plot the term premia implied by a three factor reduced form Gaussian ATSM in the top left of [Figure 3](#), which serves as a benchmark for many policy discussions (for implementation details, see for example, [Hamilton and Wu \(2012\)](#) and [Creal and Wu \(2015b\)](#)). Both the size and time variation of our estimates (bottom right) resemble the reduced form ATSM estimates.

With both time-varying risk prices and quantity of risk built in, our model does an adequate job of capturing the pattern of term premia exhibited in the data. The question is then which channel contributes more? The literature provides two opposite answers: reduced form Gaussian affine models attribute the time-varying term premia completely to a time-varying price of risk; while the literature on recursive preferences attributes it completely to a time-varying quantity of risk. Our unifying framework equips us with a more comprehensive view to answer this question. We study how much time variation there would be if we shut down one channel at a time.

First, we shut down the risk price channel by setting  $\lambda_g = 0$ , or equivalently  $\Phi_g^{Q^s} = \Phi_g$ .

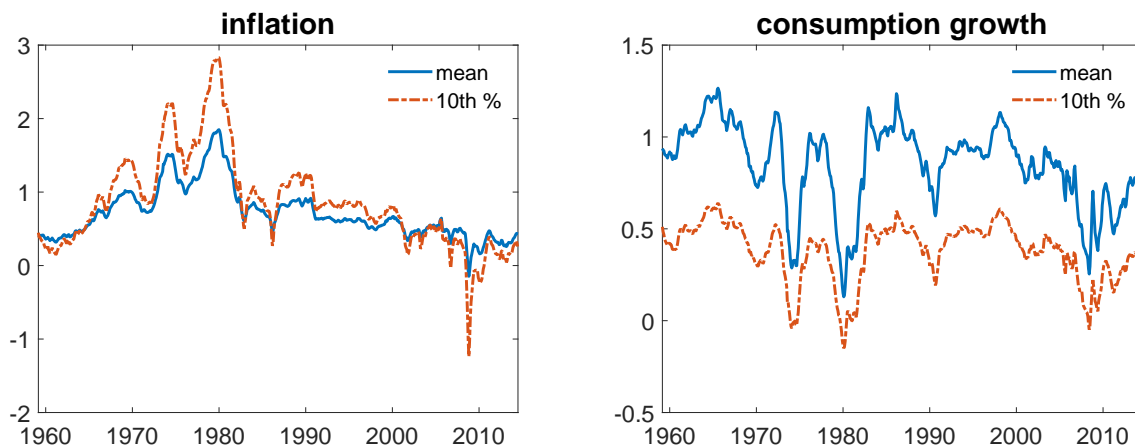
Figure 3: Estimated term premia from alternative models



*Estimated 1 and 5 year term premia from alternative models. Top: reduced-form 3 factor Gaussian ATSM. middle: SV model with no preference shock; Bottom: Gaussian model with preference shock. Blue dashed lines: 1 year; red solid lines: 5 year. Y-axis: interest rates measured in annualized percentage points.*

This model is similar to those in the long run risk literature, Bansal and Shaliastovich (2013), for example. The difference is that we model the volatility process with a non-central Gamma process guaranteeing its non-negativity, whereas the literature models it with a Gaussian process. We re-optimize the objective function subject to the constraint  $\lambda_g = 0$ , and plot the implied term premium in the bottom left of Figure 3. Without the preference shock, the term premia are essentially constant and economically insignificant. Moreover, the term

Figure 4: Five-year term premia



*Left: model implied term premium when the matrix  $\lambda_g$  is set to 0 except the diagonal entry allowing the price of expected inflation risk to load on itself. Right: model implied term premium when the matrix  $\lambda_g$  is set to 0 except the component allowing the price of expected consumption risk to load on itself.*

premia generated by this model are negative, which is the wrong sign.<sup>6</sup> A simple model with only a time-varying quantity of risk is not sufficient to account for variation in term premia.

Next, we shut down the time variation in the quantity of risk channel by setting  $h_t = 0$  in (2.11) - (2.12), but still allow a preference shock  $\lambda_g \neq 0$ . Then the factor dynamics follow a Gaussian VAR. The resulting term premia from re-optimizing this restricted model are depicted in the top right panel. Interestingly, both the size and time variation of the term premia resemble the estimates in our main model (bottom right) and the GATSM. Hence, a time-varying risk price that is a function of expected inflation and expected consumption growth generates the amount of variation of term premia as we observe from the reduced form estimates.

We have established that a time-varying risk price through the preference shock is a channel that can explain almost all of the variation in the bond term premia. We then

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<sup>6</sup>Our results are not specific to our estimates for the structural parameters. If we calibrate the structural parameters using the values from Bansal and Yaron (2004) for  $(\beta, \gamma, \eta)$ , the model still produces the same pattern.

further ask: is the price of inflation risk or consumption risk time varying? What drives the variation in this price? First, we only allow the price of expected inflation risk to vary over time, and also restrict it to comove with the expected inflation itself. We implement this by imposing the following restrictions: all components in  $\lambda_g$  are zero but the  $\lambda_{\bar{\pi},\bar{\pi}}$ , or equivalently  $\Phi_g^{\mathbb{Q}^{\$}} = \Phi_g$  for all but one component  $\phi_{\bar{\pi},\bar{\pi}}^{\mathbb{Q}^{\$}} \neq \phi_{\bar{\pi},\bar{\pi}}$ . We re-estimate the remaining parameters given this restriction. The implied five-year term premium is plotted in the left panel of [Figure 4](#), which is very similar to the estimates from [Figure 2](#). Therefore, the variation in the term premium is primarily coming from the price of expected inflation risk loading on itself, and this one single component allows us to capture the predominant variation in the term premium.

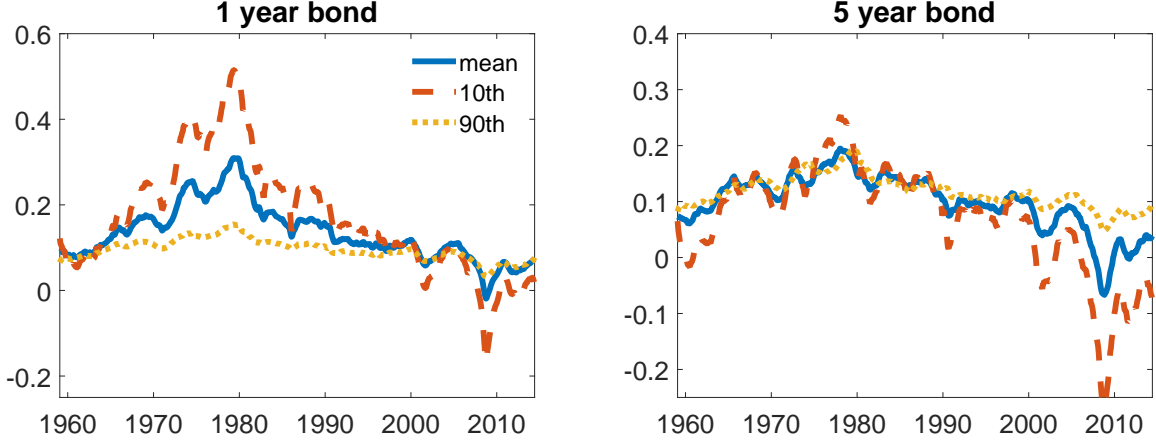
As a contrast, we plot in the right panel of [Figure 4](#) estimates of the term premia when we only allow the price of expected consumption risk to be non-zero, and to vary with itself. Although displaying as much variation, it does not resemble the key economic feature in the term premium in [Figure 2](#). For example, the term premium was lower in the 1960s, and peaked in the early 1980s in the benchmark model. Estimates in [Figure 4](#) show an opposite pattern: it was high in the middle of the 1960s, and became negative during the 1970-1980s when the term premium was generally considered to be extremely high. This is counter-intuitive.

To capture the variability of the estimated macroeconomic variables from the first stage, we plot the 10<sup>th</sup> percentile in red dashed lines. They mimic the blue lines. As we explained earlier in this section, the 90<sup>th</sup> percentile converges to an alternative local maximum. It exhibits a different economic interpretation and we do not plot it.

## 7.2 Conditional Sharpe ratios

This section studies the conditional Sharpe ratio, which is closely related to term premia. First, denote  $r_{t+1}^{(n),\$}$  as the return of buying an  $n$  period bond at time  $t$ , holding it for one

Figure 5: Estimated conditional Sharpe ratios



*Estimated conditional Sharpe ratios for log returns from the main model with preference shocks. Left: 1 year bond held for 1 month; Right: 5 year bond held for 1 month. Each graph plots the conditional Sharpe ratio when the factors  $g_t$  and  $h_t$  are estimated at their posterior mean, 10th, and 90th quantiles.*

period, and selling it at  $t + 1$  as an  $n - 1$  period bond. The excess return is

$$rx_{t+1}^{(n),\$} \equiv r_{t+1}^{(n),\$} - r_t^{\$}$$

The corresponding risk premium is

$$\mathbb{E}_t \left( rx_{t+1}^{(n),\$} \right) + \frac{1}{2} \mathbb{V}_t \left( rx_{t+1}^{(n),\$} \right) \equiv -\text{cov} \left( rx_{t+1}^{(n),\$}, m_{t+1}^{\$} \right)$$

The conditional Sharpe ratio of this asset in terms of log nominal returns is defined as

$$s_t^{(n),\$} \equiv \left[ \mathbb{E}_t \left( r_{t+1}^{(n),\$} \right) - r_t^{\$} + \frac{1}{2} \mathbb{V}_t \left( r_{t+1}^{(n),\$} \right) \right] / \sqrt{\mathbb{V}_t \left( r_{t+1}^{(n),\$} \right)} \quad (7.1)$$

where an explicit expression is available in the online appendix.

Figure 5 plots the conditional Sharpe ratio for the one year bond on the left, and five year bond on the right. The blue lines use the posterior mean estimates of the factors. We also report the conditional Sharpe ratios estimated using the 10<sup>th</sup> (red dashed lines) and 90<sup>th</sup>

percentiles (yellow dotted lines) of the MCMC draws. The Sharpe ratios increased between 1960 and 1980 from about 0.1 to the highest of 0.3 for the one year maturity, and 0.2 for the five year maturity. Then, they decrease over the second half of the sample, and became negative during the Great Recession, and end around where they started.

Both the dynamics and the level of the Sharpe ratio are sensible. The average Sharpe ratio is 0.13 for one year and 0.10 for the five year for the mean macro variables. The 10<sup>th</sup> and 90<sup>th</sup> percentiles range from 0.10 to 0.17 for the one year maturity bond, and 0.07 to 0.12 for the five year bond. Duffee (2010) shows that the Sharpe ratios in Gaussian ATSMs can be implausibly high when there are no over-identifying restrictions imposed. The restrictions in our structural model allow a lower and reasonable Sharpe ratio. In this sense, the restrictions imposed by economic theory disciplines the term structure model.

## 8 Conclusion

Two strands of related literature attribute time variation in bond term premia to two different sources: Gaussian ATSMs credit time-varying risk premia to risk prices that are functions of the conditional mean of the risk factors, whereas structural models with recursive preferences and long run risk attribute it to time-varying quantities of risk. We developed a consumption based model to capture both of these competing sources. We introduced time-varying risk prices through a preference shock that depends on current and past components of consumption growth and inflation. This generates a time-varying risk premia even when the shocks are homoskedastic. Our novel formulation of the preference shock yields analytical bond prices, gaining tractability for this class of models. We introduce a time varying quantity of risk through stochastic volatility, which follows a non-negative affine process. We found that the time-varying price of expected inflation risk driven by expected inflation itself is the primary channel empirically. On the contrary, once the preference shock is a component in the model, the presence of stochastic volatility does not alter the economic implication

of the dynamics of term premia. Moreover, a stochastic volatility model without preference shock cannot match the upward sloping unconditional nominal yield curve, the fundamental moment in the term structure. Adding the preference shock solves this problem as well.

Empirical implementation of recursive preferences requires careful attention when solving for the stochastic discount factor. A solution does not exist for certain combinations of structural parameters. Our paper provided conditions that guaranteed the existence of a solution. We make the first step to understand the parameter space, and implementing these in empirical estimation could be an interesting area of future work.

Several authors have studied term structure models with recursive preferences in DSGE models, e.g. Rudebusch and Swanson (2008), Rudebusch and Swanson (2012), van Binsbergen et al. (2012), and Dew-Becker (2014). How to introduce our technology of capturing realistic dynamics of term premia and other key aspects of bonds and other assets into a DSGE framework remains an open question, and logical next step for the literature.



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