Supplementary Material for "A Nonlinear Certainty Equivalent Approximation Method for Dynamic Stochastic Problems"

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Appendix A—Approximation

An approximation scheme approximates a function $F(\mathbf{x})$ with $\hat{F}(\mathbf{x};\mathbf{b}) = \sum_{j=0}^{n} b_j\phi_j(\mathbf{x})$ for some vector of parameters $\mathbf{b}$. A spectral method uses globally nonzero basis functions $\phi_j(\mathbf{x})$. Examples of spectral methods include ordinary or Chebyshev polynomial approximation. In contrast, a finite element method uses local basis functions where for each j the basis function $\phi_j(\mathbf{x})$ is zero except on a small part of the approximation domain. Examples of finite element methods include piecewise linear interpolation, cubic splines, and B-splines. See Cai and Judd (2014, 2015) and Judd (1998) for more details.

Chebyshev Polynomial Approximation

Chebyshev polynomials on $[-1,1]$ are defined as $\phi_j(z) = \cos(j\cos^{-1}(z))$. The Chebyshev polynomials on a general interval $[x_{\min}, x_{\max}]$ are defined as $\phi_j((2x - x_{\min} - x_{\max})/(x_{\max} - x_{\min}))$ for $j \geq 0$, and are orthogonal under the weighted inner product $\langle f, g\rangle = \int_{x_{\min}}^{x_{\max}} f(x)g(x)w(x)dx$ with the weighting function

$$w(x) = \left(1 - \left(\frac{2x - x_{\min} - x_{\max}}{x_{\max} - x_{\min}}\right)^2\right)^{-1/2}.$$

A degree D Chebyshev polynomial approximation for $V(x)$ on $[x_{\min}, x_{\max}]$ is

$$\hat{V}(x;\mathbf{b}) = \sum_{j=0}^{D} b_j\phi_j\left(\frac{2x - x_{\min} - x_{\max}}{x_{\max} - x_{\min}}\right), \tag{46}$$

where b_j are the Chebyshev coefficients.

The canonical Chebyshev nodes on $[-1,1]$ are $z_i = -\cos\left((2i-1)\pi/(2m)\right)$ for $i = 1,\ldots,m$, and the corresponding Chebyshev nodes adapted for the general interval $[x_{\min}, x_{\max}]$ are $x_i = (z_i+1)(x_{\max}-x_{\min})/2+x_{\min}$. If we have Lagrange data $\{(x_i, v_i) : i = 1,\ldots,m\}$ with $v_i = V(x_i)$, then the coefficients b_j in (46) are

$$b_j = \frac{2}{m}\sum_{i=1}^{m} v_i\phi_j(z_i), \qquad j = 1,\ldots,D, \tag{47}$$

and $b_0 = \sum_{i=1}^{m} v_i/m$. The method is called the Chebyshev regression algorithm in Judd (1998).

Multidimensional Complete Chebyshev Approximation

In a d-dimensional approximation problem, the domain of the approximation function will be

$$\left\{\mathbf{x} = (x_1,\ldots,x_d) : \, x_{\min,i} \le x_i \le x_{\max,i}, \, i = 1,\ldots d\right\},$$

Let $\mathbf{x}_{\min} = (x_{\min,1},\ldots,x_{\min,d})$ and $\mathbf{x}_{\max} = (x_{\max,1},\ldots,x_{\max,d})$. We let $[\mathbf{x}_{\min}, \mathbf{x}_{\max}]$ denote the domain. Let $\alpha = (\alpha_1,\ldots,\alpha_d)$ be a vector of nonnegative integers. Let $\phi_\alpha(\mathbf{z})$ denote the product $\prod_{i=1}^{d}\phi_{\alpha_i}(z_i)$ for $\mathbf{z} = (z_1,\ldots,z_d) \in [-1,1]^d$. Let

$$Z(\mathbf{x}) = \left(\frac{2x_1 - x_{\min,1} - x_{\max,1}}{x_{\max,1} - x_{\min,1}}, \ldots, \frac{2x_d - x_{\min,d} - x_{\max,d}}{x_{\max,d} - x_{\min,d}}\right)$$

for any $\mathbf{x} = (x_1,\ldots,x_d) \in [\mathbf{x}_{\min}, \mathbf{x}_{\max}]$. With this notation, the degree-D complete Chebyshev approximation for $V(\mathbf{x})$ is

$$\hat{V}(\mathbf{x};\mathbf{b}) = \sum_{\alpha\ge 0,\, |\alpha|\le D} b_\alpha \phi_\alpha\left(Z(\mathbf{x})\right),$$

where $|\alpha| = \sum_{i=1}^{D}\alpha_i$. This is a degree D polynomial, and has $\binom{d+D}{D}$ terms.

Appendix B—Application to a RBC model with a constraint on investment

Here we apply NLCEQ to solve a RBC model with a constraint on investment to illustrate that NLCEQ can solve problems with inequality constraints that occasionally bind (Christiano and Fisher 2000; Guerrieri and Iacoviello 2015).

Model Overview

We solve the following social planner's problem:

$$\max_{c} \mathbb{E}\left\{\sum_{t=0}^{\infty} \beta^t U(c_t)\right\} \tag{48}$$

subject to the following constraints

$$c_t + I_t = A_t k_t^{\alpha}, \tag{49}$$

$$k_{t+1} = (1-\delta)k_t + I_t, \tag{50}$$

$$I_t \geq \phi I_{ss}, \tag{51}$$

for $t \geq 0$, where c_t is consumption, I_t is investment, k_t is capital, and A_t is technology following the autoregression process

$$\ln(A_{t+1}) = \rho \ln(A_t) + \sigma \epsilon_{t+1}, \tag{52}$$

where ϵ_t is an exogenous innovation with standard normal distribution. We use the parameter values in Guerrieri and Iacoviello (2015), that is, $\beta = 0.96$, $\delta = 0.1$, $\phi = 0.975$, $\alpha = 0.33$, $\rho = 0.9$, $\sigma = 0.013$, $U(c) = (c^{1-\gamma} - 1)/(1-\gamma)$ with $\gamma = 2$. Moreover, I_{ss} is investment in the steady state of the deterministic variant of the model (48) with $A_t \equiv 1$. From the first-order conditions for the deterministic variant, we know that the steady state is

$$k_{ss} = \left(\frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right)\right)^{\frac{1}{\alpha-1}}$$

and $I_{ss} = \delta k_{ss} \approx 0.3533$. Since the value of ϕ is chosen to be close to 1, the inequality (51) will bind frequently.

Error Measure

Let $\beta^t \lambda_t$ denote the Lagrange multiplier of (51) at period t. We have the consumption Euler equation and the Kuhn-Tucker condition for (51):

$$\begin{aligned} &U'(c_t) - \lambda_t = \beta \mathbb{E}_t \left\{U'(c_{t+1}) \left(1 - \delta + \alpha A_{t+1} k_{t+1}^{\alpha-1}\right) - (1-\delta)\lambda_{t+1}\right\} \\ &\lambda_t \left(I_t - \phi I_{ss}\right) = 0 \end{aligned}$$

Similarly with the examples in Section 4, we use NLCEQ to get the estimate of the optimal consumption function, $C(k, A)$, and the function for the Lagrange multiplier, $\Lambda(k, A)$, on a domain $[0.5k_{ss}, 1.5k_{ss}] \times [0.5, 1.5]$. The optimal investment function is $\mathcal{I}(k, A) = Ak^{\alpha} - C(k, A)$, and the next-period capital is $\mathcal{K}^+(k, A) = (1-\delta)k + \mathcal{I}(k, A)$.

Using these approximate functions, for a given (K, θ), we can compute the following unit-free Euler error:

$$E_1(k, A) = \left| \frac{\beta \mathbb{E}\left\{U'(c^+) \left(1 - \delta + \alpha A^+ \left(k^+\right)^{\alpha-1}\right) - (1-\delta)\lambda^+\right\} + \lambda}{U'(c)} - 1 \right|, \tag{53}$$

where A^+ is the next-period productivity, $c = C(k, A)$, $\lambda = \Lambda(k, A)$, $k^+ = \mathcal{K}^+(k, A)$, $c^+ = C(k^+, A^+)$, and $\lambda^+ = \Lambda(k^+, A^+)$. We use the 15-point Gauss-Hermite quadrature rule to estimate the integration in (53). Similarly, the unit-free error for the Kuhn-Tucker condition is

$$E_2(k, A) = \left| \lambda \left(\frac{I}{\phi I_{ss}} - 1 \right) \right|$$

with $I = \mathcal{I}(k, A)$. The error measure for the investment constraint (51) cannot be omitted, because the true solution of the model without the constraint (51) will also have $E_1(k, A) = 0$ and $E_2(k, A) = 0$ with $\lambda = 0$, that is, E_1 and E_2 are not enough for error measurement. Thus we need to check the following unit-free error

$$E_3(k, A) = \max \left(0,\ 1 - \frac{I}{\phi I_{ss}} \right).$$

We then compute the following global $\mathcal{L}^\infty$ and $\mathcal{L}^1$ errors on a set of points (k, A), denoted $\mathscr{D}$, to measure the accuracy of our solution:

$$\mathcal{E}_{\mathcal{L}^\infty} = \max_{i=1,2,3} \left\{ \max_{(k,A)\in\mathscr{D}} E_i(k, A) \right\},$$

$$\mathcal{E}_{\mathcal{L}^1} = \max_{i=1,2,3} \left\{ \frac{1}{|\mathscr{D}|} \sum_{(k,A)\in\mathscr{D}} E_i(k, A) \right\},$$

where $|\mathscr{D}|$ is the number of points in the set $\mathscr{D}$. We choose two sets of points, $\mathscr{D}_1$ and $\mathscr{D}_2$, where $\mathscr{D}_1$ is a set of 10,000 randomly and uniformly drawn in $[0.7k_{ss}, 1.3k_{ss}] \times [0.7, 1.3]$,[22] and $\mathscr{D}_2$ is a set of 10,000 simulated points in the path of (k_t, A_t), where $k_0 = k_{ss}$, $A_0 = 1$, A_{t+1} is simulated based on the

[22]Guerrieri and Iacoviello (2015) show their results in a much narrower range for A, $[0.97, 1.025]$. However, our range for A, $[0.7, 1.3]$, is reasonable: from $\ln(A_{t+1}) = \rho \ln(A_t) + \sigma\epsilon_{t+1}$, if A_t is inside the following range

$$\left[\exp\left(\frac{-2\sigma}{1-\rho} \right), \exp\left(\frac{2\sigma}{1-\rho} \right) \right],$$

which is close to $[0.7, 1.3]$, then only when ϵ_{t+1} are always simulated in $[-2, 2]$, we can make sure that A_{t+1} is inside the same range. That is, if A_t is at one end of the range, then it has about 2.3% probability that A_{t+1} is outside the range.

Table 7: Errors of the NLCEQ solution with degree-D complete Chebyshev polynomials for the RBC model with a constraint on investment

D	Approx Error for c		Approx Error for λ		Global Error on $\mathscr{D}_1$		Global Error on $\mathscr{D}_2$	
	$\widehat{E}_{\mathcal{L}^\infty}$	$\widehat{E}_{\mathcal{L}^1}$	$\widehat{E}_{\mathcal{L}^\infty}$	$\widehat{E}_{\mathcal{L}^1}$	$\mathcal{E}_{\mathcal{L}^\infty}$	$\mathcal{E}_{\mathcal{L}^1}$	$\mathcal{E}_{\mathcal{L}^\infty}$	$\mathcal{E}_{\mathcal{L}^1}$
10	5.2(−3)	1.7(−3)	2.5(−2)	5.4(−3)	2.6(−2)	3.1(−3)	2.1(−2)	2.5(−3)
20	2.6(−3)	5.0(−4)	1.5(−2)	1.4(−3)	1.3(−2)	7.9(−4)	5.3(−3)	6.9(−4)
50	1.6(−3)	8.5(−5)	9.8(−3)	2.7(−4)	4.5(−3)	1.4(−4)	4.9(−4)	9.7(−5)
100	8.2(−4)	2.1(−5)	2.2(−3)	6.9(−5)	2.0(−3)	1.2(−4)	1.9(−3)	1.8(−4)

Note: $\zeta(-j)$ means $\zeta \times 10^{-j}$.

stochastic process (52), and $k_{t+1} = \mathcal{K}^{+}(k_t, A_t)$ for $t = 0, ..., 9999$. Thus, $\mathscr{D}_2$ represents the ergodic set of (k, A), so the errors on $\mathscr{D}_2$ are weighted errors with more weights on the area around the steady state.

Numerical Results

In the transformation step of the NLCEQ method, we choose $T = 100$ and the problem becomes

$$\widetilde{V}\left(k_0, A_0\right) = \max_{c} \sum_{t=0}^{T-1} \beta^t U\left(c_t\right) + \beta^T \widetilde{V}_T\left(k_T, A_T\right), \tag{54}$$

subject to (49)-(51) with a deterministic process of A_t: $\ln(A_{t+1}) = \rho \ln(A_t)$. The terminal value function $\widetilde{V}_T\left(k, A\right)$ is given as $U(0.7Ak^{\alpha})/(1-\beta)$. In the approximation step of NLCEQ, we use the tensor grid of Chebyshev nodes ($D+1$ nodes in each dimension) and degree-D complete Chebyshev polynomials.

Table 7 reports approximation errors and global errors of the solution of NLCEQ over two sets of points, $\mathscr{D}_1$ and $\mathscr{D}_2$, for various degrees D. We see that higher degree approximation achieves higher accuracy, and the weighted errors on $\mathscr{D}_2$ are a bit smaller than those on $\mathscr{D}_1$. Because of the kinks caused by the frequently binding constraint on investment, a polynomial approximation is not very good at approximating functions with kinks until a high degree approximation (this is reflected by the approximation errors of Lagrange multiplier λ in the table, moreover most of global errors in the table come from the investment constraint error E_3 because of the kinks on the investment function), so NLCEQ achieves accuracy with $O(10^{-3})$ in $\mathcal{L}^\infty$ or $O(10^{-4})$ in $\mathcal{L}^1$ until the degree-50 approximation.[23]

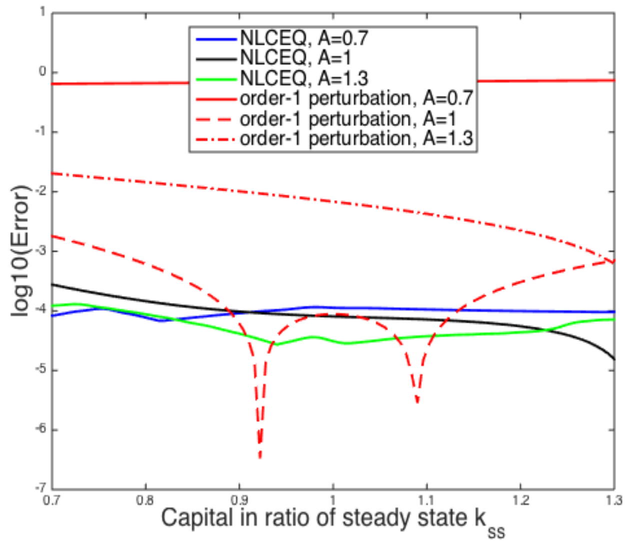
However, the order-1 perturbation (log-linearization) method has an $\mathcal{L}^\infty$ global error up to 0.73 and an $\mathcal{L}^1$ global error up to 0.17 on the domain $[0.7k_{ss}, 1.3k_{ss}] \times [0.7, 1.3]$, although its $\mathcal{L}^\infty$ error is 0.02 and $\mathcal{L}^1$ error is 0.003 for the model without the investment constraint (51). The order-2 perturbation method does not improve the accuracy as its $\mathcal{L}^\infty$ error is 0.8 and $\mathcal{L}^1$ error is 0.18., although it increases about two order accuracy for the model without the investment constraint (51). Therefore, this shows that NLCEQ is much more accurate, about two or three orders of magnitude higher, than the order-1 and order-2 perturbation methods for this problem with the occasionally binding constraint.

The comparison between NLCEQ and log-linearization is also shown in Figure 5, which shows the global errors of their solutions when $A = 0.7$, 1, and 1.3. The NLCEQ solution is the one with degree-100 complete Chebyshev polynomial approximation. Figure 5 shows clearly that NLCEQ is much more accurate than log-linearization globally, particularly when the state is not close to the steady state.

We now try piecewise bilinear interpolation as the approximation method, because piecewise bilinear interpolation can deal with the kinks better than polynomials. For the approximation nodes, we choose the tensor grid of n equally spaced capital in $[0.5k_{ss}, 1.5k_{ss}]$ and n equally spaced productivity in $[0.5, 1.5]$. Table 8 lists approximation errors and global errors from NLCEQ with piecewise bilinear interpolation, and we found that the piecewise bilinear interpolation has smaller, about one order of magnitude, errors than

[23]We also tried the case with $\phi = 0$, and found that its NLCEQ solution has a bit smaller errors than those in Table 7.

Figure 5: Errors of the solutions from NLCEQ or log-linearization for the RBC model with a constraint on investment



the complete Chebyshev polynomials when they use the same number of approximation nodes.

Table 9 shows global errors for various standard deviation σ (approximation errors are independent on σ). We see that a smaller σ has smaller errors and it has about four-digit accuracy for the smallest $\sigma = 0.001$. When $\sigma = 0.05$, the errors are up to $O(10^{-3})$ and there are almost no improvement by increasing n from 51 to 101. Moreover, when σ is up to 0.05, the global

Table 8: Errors of the NLCEQ solution with piecewise bilinear interpolation for the RBC model with a constraint on investment

n	Approx Error for c		Approx Error for λ		Global Error on $\mathscr{D}_1$		Global Error on $\mathscr{D}_2$	
	$\widehat{E}_{\mathcal{L}^\infty}$	$\widehat{E}_{\mathcal{L}^1}$	$\widehat{E}_{\mathcal{L}^\infty}$	$\widehat{E}_{\mathcal{L}^1}$	$\mathcal{E}_{\mathcal{L}^\infty}$	$\mathcal{E}_{\mathcal{L}^1}$	$\mathcal{E}_{\mathcal{L}^\infty}$	$\mathcal{E}_{\mathcal{L}^1}$
21	3.7(−3)	1.1(−4)	4.0(−2)	3.6(−3)	5.8(−3)	7.6(−4)	1.7(−3)	3.1(−4)
51	1.9(−3)	2.6(−5)	7.3(−3)	5.9(−4)	8.7(−4)	1.7(−4)	4.5(−4)	1.1(−4)
101	7.5(−4)	4.1(−6)	4.7(−3)	1.4(−4)	3.6(−4)	1.1(−4)	2.5(−4)	9.8(−5)

Note: $\zeta(-j)$ means $\zeta \times 10^{-j}$.

Table 9: Errors of the NLCEQ solution with piecewise bilinear interpolation for the RBC model with a constraint on investment and various standard deviations

σ	n	Global Error on $\mathscr{D}_1$		Global Error on $\mathscr{D}_2$	
		$\mathcal{E}_{\mathcal{L}^\infty}$	$\mathcal{E}_{\mathcal{L}^1}$	$\mathcal{E}_{\mathcal{L}^\infty}$	$\mathcal{E}_{\mathcal{L}^1}$
0.001	21	7.5(−3)	8.4(−4)	2.5(−4)	3.5(−5)
	51	9.3(−4)	1.2(−4)	4.5(−5)	8.2(−6)
	101	2.9(−4)	3.1(−5)	3.0(−6)	6.8(−7)
0.02	21	5.6(−3)	7.9(−4)	2.0(−3)	4.6(−4)
	51	1.3(−3)	2.8(−4)	7.9(−4)	2.4(−4)
	101	6.6(−4)	2.4(−4)	4.2(−4)	2.3(−4)
0.05	21	8.1(−3)	1.8(−3)	9.4(−3)	1.5(−3)
	51	2.9(−3)	1.4(−3)	5.9(−3)	1.4(−3)
	101	2.8(−3)	1.4(−3)	4.0(−3)	1.3(−3)

Note: $\zeta(-j)$ means $\zeta \times 10^{-j}$.

Table 10: Errors of the NLCEQ solution using $\ln(A_{t+1}) = \rho \ln(A_t) - 0.5\sigma^2$

σ	n	Global Error on $\mathscr{D}_1$		Global Error on $\mathscr{D}_2$	
		$\mathcal{E}_{\mathcal{L}^\infty}$	$\mathcal{E}_{\mathcal{L}^1}$	$\mathcal{E}_{\mathcal{L}^\infty}$	$\mathcal{E}_{\mathcal{L}^1}$
0.05	21	5.8(−3)	9.5(−4)	7.5(−3)	7.8(−4)
	51	1.8(−3)	5.1(−4)	4.9(−3)	4.6(−4)
	101	1.7(−3)	4.7(−4)	3.5(−3)	3.9(−4)

Note: $\zeta(-j)$ means $\zeta \times 10^{-j}$.

errors on the ergodic set $\mathscr{D}_2$ are bigger than those on $\mathscr{D}_1$, because the domain containing $\mathscr{D}_1$, $[0.7k_{ss}, 1.3k_{ss}] \times [0.7, 1.3]$, is not large enough to contain $\mathscr{D}_2$ for large σ.

However, the errors for large σ can be decreased by changing the deterministic transition law of A_t to $\ln(A_{t+1}) = \rho \ln(A_t) - 0.5\sigma^2$. Table 10 shows errors for $\sigma = 0.05$ using the new deterministic transition law of A_t and piecewise bilinear interpolation. We see that the errors are smaller than those in 9 from $\ln(A_{t+1}) = \rho \ln(A_t)$. Moreover, a larger n clearly improves the accuracy of the solution.

Since global errors cannot represent true errors compared with the true

Table 11: “True” relative errors of the NLCEQ solution for the RBC model with a constraint on investment

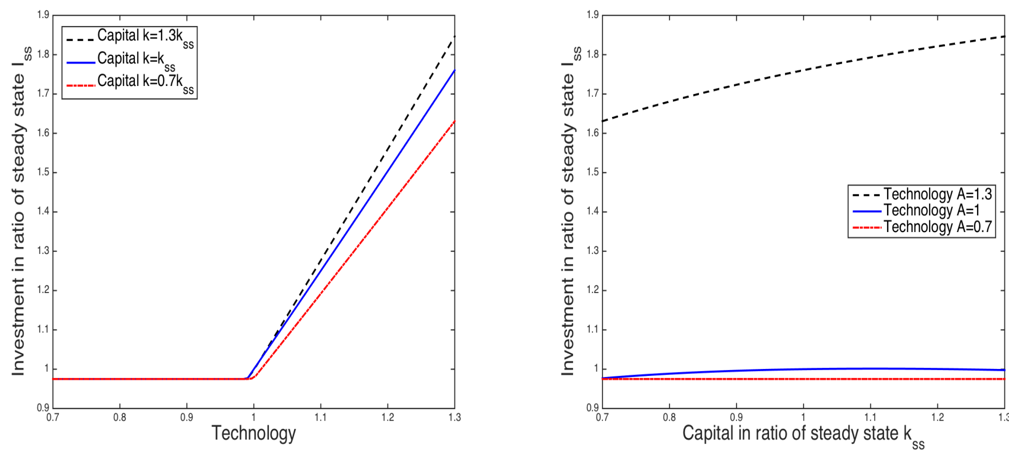
n	Piecewise Bilinear Interp.		Complete Chebyshev Poly.	
	error in $\mathcal{L}^\infty$	error in $\mathcal{L}^1$	error in $\mathcal{L}^\infty$	error in $\mathcal{L}^1$
21	6.0(−3)	3.2(−4)	5.2(−3)	1.3(−3)
51	3.2(−3)	1.5(−4)	2.7(−3)	2.5(−4)
101	4.2(−4)	1.1(−5)	1.3(−3)	1.5(−4)

Note: $\zeta(-j)$ means $\zeta \times 10^{-j}$.

solution, we implement shape-preserving value function iteration with rational spline interpolation (Cai and Judd 2012) to derive the “true” solution and then check the “true” errors. We follow Tauchen (1986) to approximate the process of $\ln(A_t)$ with a Markov chain of 101 equally spaced values in $[0.5, 1.5]$, and use 101 equally spaced nodes for capital in $[0.5k_{ss}, 1.5k_{ss}]$ as the approximation nodes for the rational spline interpolation for each discrete value of the Markov process $\ln(A_t)$. The value function iteration stops while the relative change of two consecutive value functions is less than 10^{-6}. With these converged “true” solution, Table 11 reports “true” relative errors for consumption function in the domain of k and A, $[0.7k_{ss}, 1.3k_{ss}] \times [0.7, 1.3]$, from NLCEQ with degree-$(n-1)$ complete Chebyshev polynomials or piecewise bilinear interpolation with $n \times n$ approximation nodes. We see that these errors are close to those global errors in Table 7 or Table 9. We also see that the “true” relative errors from piecewise bilinear interpolation are smaller than those from complete Chebyshev polynomials when $n = 101$.

Figure 6 shows the optimal investment policy functions from NLCEQ with piecewise bilinear interpolation ($n = 101$). We see that when technology $A_t > 1$ and capital $k_t > 0.7k_{ss}$, the investment is always bigger than its lower bound. But if A_t is small then the investment is binding at the lower bound.

Figure 6: Investment policy function for the RBC model with a constraint on investment



Appendix C—Equilibrium Conditions in the New Keynesian DSGE Model

The final-good firm buys intermediate goods $y_{i,t}$ from intermediate firms to produce a final good y_t with the following production function

$$y_t = \left(\int_0^1 y_{i,t}^{\frac{\alpha-1}{\alpha}} di \right)^{\frac{\alpha}{\alpha-1}} \tag{55}$$

then sell y_t at a price p_t. Let $p_{i,t}$ be prices of $y_{i,t}$, then the final-good firm chooses $y_{i,t}$ to maximize its profit:

$$\max_{y_{i,t}} p_t y_t - \int_0^1 p_{i,t} y_{i,t} di.$$

Its first-order condition implies

$$y_{i,t} = y_t \left(\frac{p_{i,t}}{p_t} \right)^{-\alpha}. \tag{56}$$

The intermediate firms rent labor supply $\ell_{i,t}$ from the household with a wage rate w_t and produce $y_{i,t}$ with a simple production function

$$y_{i,t} = \ell_{i,t}, \tag{57}$$

and sell $y_{i,t}$ at a price $p_{i,t}$ to the final-good firm. The intermediate firms are assumed to have Calvo-type prices: a fraction $1-\theta$ of the firms have optimal prices and the remaining fraction θ of the firms keep the same price as in the previous period.

A re-optimizing intermediate firm $i \in [0,1]$ chooses its price $p_{i,t}$ to maximize the current value of profit over the time when the optimal $p_{i,t}$ remains effective:

$$\max_{p_{i,t}} \mathbb{E}_t \left\{ \sum_{j=0}^{\infty} \left(\prod_{k=0}^{j} \beta_{t+k} \right) \lambda_{t+j} \theta^j \left(p_{i,t} y_{i,t+j} - w_{t+j} \ell_{i,t+j} \right) \right\} \tag{58}$$

subject to the constraints $y_{i,t+j} = \ell_{i,t+j}$ from (57) and

$$y_{i,t+j} = y_{t+j} \left(\frac{p_{i,t}}{p_{t+j}} \right)^{-\alpha}$$

from (56) by letting $p_{i,t+j} = p_{i,t}$. Here λ_t is the Lagrange multiplier of the budget constraint (34). From the first-oder conditions of the household problem (35), λ_t satisfies the following equation:

$$\lambda_t = \frac{1}{p_t c_t}. \tag{59}$$

The first-order condition of the re-optimizing intermediate firm problem (58) implies

$$\mathbb{E}_t \left\{ \sum_{j=0}^{\infty} \left(\prod_{k=0}^{j} \beta_{t+k} \right) \lambda_{t+j} \theta^j p_{t+j}^{\alpha} y_{t+j} \left(p_{i,t} - \frac{\alpha}{\alpha-1} w_{t+j} \right) \right\} = 0 \tag{60}$$

Let $\pi_{t,j} = p_{t+j}/p_t$. From (38), (59) and (60), for any re-optimizing firm i we have

$$\frac{p_{i,t}}{p_t} \equiv q_t = \frac{\alpha \chi_{t,1}}{(\alpha-1)\chi_{t,2}} \tag{61}$$

where

$$\begin{aligned}
\chi_{t,1} &\equiv y_t \ell_t^{\eta} + \mathbb{E}_t \left\{ \sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} \beta_{t+k} \right) \theta^j \pi_{t,j}^{\alpha} y_{t+j} \ell_{t+j}^{\eta} \right\} \\
\chi_{t,2} &\equiv \frac{y_t}{c_t} + \mathbb{E}_t \left\{ \sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} \beta_{t+k} \right) \theta^j \pi_{t,j}^{\alpha-1} \frac{y_{t+j}}{c_{t+j}} \right\}
\end{aligned}$$

We have the recursive formulas for $\chi_{t,1}$ and $\chi_{t,2}$:

$$\chi_{t,1} = y_t \ell_t^{\eta} + \theta \mathbb{E}_t \left\{ \beta_{t+1} \pi_{t+1}^{\alpha} \chi_{t+1,1} \right\} \tag{62}$$

$$\chi_{t,2} = \frac{y_t}{c_t} + \theta \mathbb{E}_t \left\{ \beta_{t+1} \pi_{t+1}^{\alpha-1} \chi_{t+1,2} \right\} \tag{63}$$

From (55) and (56), we have

$$\begin{aligned}
p_t &= \left(\int_0^1 p_{i,t}^{1-\alpha} di \right)^{\frac{1}{1-\alpha}} \\
&= \left((1-\theta)(q_t p_t)^{1-\alpha} + \theta \int_0^1 p_{i,t-1}^{1-\alpha} di \right)^{\frac{1}{1-\alpha}} \\
&= \left((1-\theta)(q_t p_t)^{1-\alpha} + \theta p_{t-1}^{1-\alpha} \right)^{\frac{1}{1-\alpha}}
\end{aligned}$$

as

$$p_{t-1} = \left(\int_0^1 p_{i,t-1}^{1-\alpha} di \right)^{\frac{1}{1-\alpha}}$$

This follows that

$$q_t = \left(\frac{1 - \theta \pi_t^{\alpha-1}}{1-\theta} \right)^{\frac{1}{1-\alpha}} \tag{64}$$

From (56), (57) and the following market clearing condition

$$\ell_t = \int_0^1 \ell_{i,t} di,$$

we get

$$\begin{aligned} v_{t+1} \equiv \ell_t / y_t &= \int_0^1 \left(\frac{p_{i,t}}{p_t} \right)^{-\alpha} di \\ &= (1-\theta) q_t^{-\alpha} + \theta \int_0^1 \left(\frac{p_{i,t-1}}{p_t} \right)^{-\alpha} di \\ &= (1-\theta) q_t^{-\alpha} + \theta \pi_t^{\alpha} \int_0^1 \left(\frac{p_{i,t-1}}{p_{t-1}} \right)^{-\alpha} di \\ &= (1-\theta) q_t^{-\alpha} + \theta \pi_t^{\alpha} v_t \end{aligned} \tag{65}$$

Appendix D—Steady State of the New Keynesian DSGE Model

From (63), the steady state of $\chi_{t,2}$ is

$$\chi_2^* = \frac{1}{(1-s_g)(1-\theta\beta^*(\pi^*)^{\alpha-1})}$$

with the given $\pi^* = 1.005$. From (61) and (64), the steady state of $\chi_{t,1}$ is

$$\chi_1^* = \chi_2^* q^* \frac{\alpha - 1}{\alpha}$$

with

$$q^* = \left(\frac{1 - \theta\,(\pi^*)^{\alpha-1}}{1-\theta} \right)^{\frac{1}{1-\alpha}}$$

and from (65) the steady state of v_t is

$$v^* = \frac{(1-\theta)(q^*)^{-\alpha}}{1-\theta(\pi^*)^{\alpha}}$$

Therefore, from $v_t = \ell_t / y_t$ and (62), we get

$$y^* = \left(\frac{\chi_1^* \left(1-\theta\beta^* (\pi^*)^{\alpha}\right)}{(v^*)^{\eta}} \right)^{\frac{1}{1+\eta}}$$