Recursive utility using the stochastic maximum principle

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Abstract

Motivated by the problems of the conventional model in rationalizing market data, we derive the equilibrium interest rate and risk premiums using recursive utility in a continuous time model. We use the stochastic maximum principle to analyze the model. This method uses forward/backward stochastic differential equations, and works when the economy is not Markovian, which can be the case with recursive utility. With existence granted, the wealth portfolio is characterized in equilibrium in terms of utility and aggregate consumption. The equilibrium real interest rate is derived, and the resulting model is shown to be consistent with reasonable values of the parameters of the utility function when calibrated to market data, under various assumptions.

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JEL-Code: G10, G12, D9, D51, D53, D90, E21.

1 Introduction

Rational expectations, a cornerstone of modern economics and finance, has been under attack for quite some time. Questions like the following were sometimes asked: Are asset prices too volatile relative to the information arriving in the market? Is the mean risk premium on equities over the riskless

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rate too large? Is the real interest rate too low? Is the market's risk aversion too high?

Mehra and Prescott (1985) raised some of these questions in their well-known paper, using a variation of Lucas's (1978) pure exchange economy with a Kydland and Prescott (1982) "calibration" exercise. They chose the parameters of the endowment process to match the sample mean, variance and the annual growth rate of per capita consumption in the years 1889 - 1978. The puzzle is that they were unable to find a plausible parameter pair of the utility discount rate and the relative risk aversion to match the sample mean of the annual real rate of interest and of the equity premium over the 90-year period.

The puzzle has been verified by many others, e.g., Hansen and Singleton (1983), Ferson (1983), Grossman, Melino, and Shiller (1987). Many theories have been suggested during the years to explain the puzzle[1].

We reconsider recursive utility in continuous time along the lines of Duffie and Epstein (1992a-b). In their papers two ordinally equivalent versions of recursive utility were established, and one version was analyzed by the use of dynamic programming. The version left out is analyzed in the present paper. Our method is the stochastic maximum principle, which gives explicit results for both risk premiums and the short rate. This method does not require the underlying processes to be Markov. This may be important in applications. For example, in Bollerslev, Engle, and Wooldridge (1988) it is indicated that the conditional variance of the market return fluctuates across time. When the conditional variance is random, the state price deflator is not a Markov process but still our approach is valid. With random conditional moments, dynamic programming may not be appropriate, which follows from the nature of the Bellman equation.

When evaluating utility of consumption, the recursive utility maximizer is not myopic, but rather takes into account more than just the present. As a consequence, when calculating the conditional probabilities of the future state prices of the economy, not only the present, but also the past values of the basic economic variables matter, i.e., the Markov property can fail[2]. The

[1]Constantinides (1990) introduced habit persistence in the preferences of the agents. Also Campbell and Cochrane (1999) used habit formation. Rietz (1988) introduced financial catastrophes, Barro (2005) developed this further, Weil (1992) introduced non-diversifiable background risk, and Heaton and Lucas (1996) introduce transaction costs. There is a rather long list of other approaches aimed to solve the puzzles, among them are borrowing constraints (Constantinides et al. (2001)), taxes (Mc Grattan and Prescott (2003)), loss aversion (Benartzi and Thaler (1995)), survivorship bias (Brown, Goetzmann and Ross (1995)), and heavy tails and parameter uncertainty (Weitzmann (2007)).

[2]Our model does not violate history independence in the sense of Section 6 of Kreps and Porteus (1978).

conditional distribution of of future consumption may depend on history in complicated ways. The stochastic maximum principle allows us to derive the optimality conditions without explicitly specifying the dependence.

We base our treatment on the basic framework developed by Duffie and Epstein (1992a-b) and Duffie and Skiadas (1994), which elaborate the foundational work by Kreps and Porteus (1978) and Epstein and Zin (1989) of recursive utility in dynamic models. The data set we use to calibrate the model is the same as the one used by Mehra and Prescott (1985) in their seminal paper on this subject.

Generally not all income is investment income. We assume that one can view exogenous income streams as dividends of some shadow asset, in which case our model is valid if the market portfolio is expanded to include new assets. In reality the latter are not traded, so the return to the wealth portfolio is not readily observable or estimable from available data. We indicate how the model may be adjusted to account for this under various assumptions, when the market portfolio is not a proxy for the wealth portfolio. Here we also present an example using Norwegian data from the period 1971-2014, in which case we do have the summary statistics related to the wealth portfolio. The present model calibrates very well to these data.

Besides giving new insights about these interconnected puzzles, the recursive model is likely to lead to many other results that are difficult, or impossible, to obtain using, for example, the conventional, time additive and separable expected utility model. One example included in the paper is related to empirical regularities for Government bills.[3]

Some of extant literature contributes to more realistic, but also more complex models, often based on approximations. An example is Bansal and Yaron (2004) exploring 'long run consumption risk'. For a relative risk aversion of 10 and EIS of 1.5, they are able to replicate the stylized facts quite well. They use the Campbell and Shiller (1988) approximation for the log interest rate. Their work is based on the Epstein and Zin (1989) discrete time approach, in which they employ a richer economic environment. Not surprisingly, this paper comes a long way in explaining several asset pricing anomalies. In contrast, our expression for the equilibrium short rate is exact,

[3]There is by now a long standing literature that has been utilizing recursive preferences. We mention Avramov and Hore (2007), Avramov et al. (2010), Eraker and Shaliastovich (2009), Hansen, Heaton, Lee, Roussanov (2007), Hansen, Heaton, Lee (2008), Hansen and Scheinkman (2009), Wacther (2012), Campbell (1996), Bansal and Yaron (2004), Kocherlakota (1990b), and Ai (2010) to name some important contributions. Related work is also in Browning et. al. (1999), on consumption see Attanasio (1999), on climate risk see Cai, Judd, and Lontzek (2013, 2015), and Pindyck and Wang (2013). Bansal and Yaron (2004) study a richer economic environment than we employ.

and so is the expression for the risk premiums. Using our approach with a less elaborate model, we are able to explain many of the same features, for more plausible values of the preference parameters.

In particular, also our model predicts lower asset prices as a result of a rise in consumption volatility. Furthermore, when the EIS is larger than 1, agents demand a large equity premium because they fear that a reduction in economic growth prospects or a rise in economic uncertainty will lower asset prices. As noticed by Bansal and Yaron (2004), this can justify many of the observed features of asset market data from a quantitative point of view.

In order to address the particular puzzle at hand, it is a clear advantage to deviate as little as possible from the basic framework in which it was discovered. This way one obtains a laboratory effect, where it is possible to learn what really makes the difference. Otherwise it is easy to get lost in an ever increasing and complex model framework. From our approach it follows that the solution is simply the new preferences. We do not even need unspecified "factors" in the model of the financial market (as Duffie Epstein (1992a) use).

It has been a goal in the modern theory of asset pricing to internalize probability distributions of financial assets. To a large extent this has been achieved in our approach. Consider the logic of the Lucas-style models. Aggregate consumption is a given diffusion process. The solution of a system of forward/backward stochastic differential equations (FBSDE) provide the main characteristics in the probability distributions of future utility. With existence of a solution to the FBSDE granted, market clearing finally determines the characteristics in the wealth portfolio from the corresponding characteristics of the utility and aggregate consumption processes.

The paper is organized as follows: Section 2 starts with a brief introduction to recursive utility in continuous time, in Section 3 we derive the first order conditions, Section 4 details the financial market, in Section 5 we analyze our chosen version of recursive utility. In Section 6 we summarize the main results, and present some calibrations. Section 7 explores various alternatives when the market portfolio is not a proxy for the wealth portfolio, Section 8 presents the calibration to Norwegian data, Section 9 points out some extensions, and Section 10 concludes.

2 Recursive Stochastic Differentiable Utility

In this section we recall the essentials of recursive, stochastic, differentiable utility along the lines of Duffie and Epstein (1992a-b) and Duffie and Skiadas (1994).

We are given a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, t \in [0,T], P)$ satisfying the 'usual' conditions, and a standard model for the stock market with Brownian motion driven uncertainty, N risky securities and one risk-less asset (Section 5 provides more details). Consumption processes are chosen form the space L of square integrable progressively measurable processes with values in $\mathbb{R}_+$.

The stochastic differential utility $U : L \to \mathbb{R}$ is defined as follows by two primitive functions: $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $A : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ is the real line.

The function $f(t, c_t, V_t)$ corresponds to a felicity index at time t, and A corresponds to a measure of absolute risk aversion (of the Arrow-Pratt type) for the agent. In addition to current consumption c_t, the function f depends on utility V_t, and it may also depend on time t as well as the state of the world $\omega \in \Omega$.

The utility process V for a given consumption process c, satisfying $U(c) = V_0$ and $V_T = 0$, is given by the representation

$$V_t = E_t\Big\{ \int_t^T \big(f(t, c_s, V_s) - \frac{1}{2} A(V_s)\, Z_s' Z_s\big) ds \Big\}, \quad t \in [0,T] \tag{1}$$

where $E_t(\cdot)$ denotes conditional expectation given $\mathcal{F}_t$ and Z_t is an $\mathbb{R}^d$-valued square-integrable progressively measurable volatility process, to be determined in our analysis. Prime means transpose. The term $Z_t' Z_t dt = d[V,V]_t$ where $[V,V]_t$ is the quadratic variation of the of V. Here d is the dimension of the Brownian motion B_t. V_t is the remaining utility for c at time t, conditional on current information $\mathcal{F}_t$, and $A(V_t)$ is penalizing for risk [4].

Recall the *time-less* situation with a mean zero risk X having variance σ^2, where the certainty equivalent m is defined by $Eu(w + X) := u(w - m)$ for a constant wealth w. Then the Arrow-Pratt approximation to m, valid for "small" risks, is given by $m \approx \frac{1}{2} A(w)\sigma^2$, where $A(\cdot)$ is the absolute risk aversion associated with u. We would expect this analogy to work well in a continuous-time model with Brownian driven uncertainty.

If, for each consumption process c_t, there is a well-defined utility process V, the stochastic differential utility U is defined by $U(c) = V_0$, the initial utility. The pair (f, A) generating V is called an aggregator.

Since $V_T = 0$ and $\int Z_t dB_t$ is is assumed to be a martingale, (1) has the stochastic differential equation representation

$$dV_t = \Big(- f(t, c_t, V_t) + \frac{1}{2} A(V_t)\, Z_t' Z_t \Big)\, dt + Z_t' dB_t. \tag{2}$$

If terminal utility different from zero is of interest, like for applications to e.g., life insurance, then V_T may be different from zero. U is monotonic and

[4] Our sign convention for A is minus the one used in Duffie and Epstein (1992a,b).

risk averse if $A(\cdot) \geq 0$ and f is jointly concave and increasing in consumption. A may also depend on time t.

The preference ordering represented by recursive utility is assumed to satisfy: Dynamic consistency, in the sense of Johnsen and Donaldson (1985); Independence of past consumption; and State independence of time preference (see Skiadas (2009a)).

Unlike expected utility theory in a *timeless* situation, i.e., when consumption only takes place at the end, in a *temporal* setting where the agent consumes in every period, derived preferences do not satisfy the substitution axiom (e.g., Mossin (1969), Kreps (1988)). Thus additive Eu-theory in a dynamic, temporal context has no axiomatic underpinning, unlike recursive utility (see Kreps and Porteus (1985), Chew and Epstein (1991)). It is notable that one of the four central axioms in the latter reference, *recursivity*, is essentially identical to the notion of consistency the sense of Johnsen and Donaldson (1985).

In this paper we consider the following specification: It is know as the Kreps-Porteus utility representation[5], which corresponds to the aggregator of the form

$$f_1(c,v) = \frac{\delta}{1-\rho}\frac{c^{1-\rho}-v^{1-\rho}}{v^{-\rho}} \quad \text{and} \quad A_1(v) = \frac{\gamma}{v} \tag{3}$$

If, for example, $A_1(v) = 0$ for all v, this means that the recursive utility agent is risk neutral. This is the main version that we analyze.

The parameters are assumed to satisfy $\rho \geq 0, \rho \neq 1, \delta \geq 0, \gamma \geq 0, \gamma \neq 1$ (when $\rho = 1$ or $\gamma = 1$ logarithms apply). The elasticity of intertemporal substitution in consumption is denoted by $\psi = 1/\rho$. The parameter ρ we refer to as the *time* preference parameter. The version (3) yields the desired disentangling of γ from ρ.

An ordinally equivalent specification can be derived as follows. When the aggregator (f_1, A_1) is given corresponding to the utility function U_1, there exists a strictly increasing and smooth function $\varphi(\cdot)$ such that the ordinally equivalent $U_2 = \varphi \circ U_1$ has the aggregator (f_2, A_2) where

$$f_2(c,v) = ((1-\gamma)v)^{-\frac{\gamma}{1-\gamma}} f_1(c, ((1-\gamma)v)^{\frac{1}{1-\gamma}}), \qquad A_2 = 0.$$

The function φ is given by

$$U_2 = \frac{1}{1-\gamma} U_1^{1-\gamma}, \tag{4}$$

[5]If the certainty equivalent is obtained by expected utility, preferences fall into the Kreps and Porteus (1978) family. This is the continuous-time limit of the CES specification examined in discrete time by Epstein and Zin (1989).

for the Kreps-Porteus specification. It has the CES-form

$$f_2(c,v) = \frac{\delta}{1-\rho}\,\frac{c^{1-\rho} - ((1-\gamma)v)^{\frac{1-\rho}{1-\gamma}}}{((1-\gamma)v)^{\frac{1-\rho}{1-\gamma}-1}}, \qquad A_2(v) = 0. \tag{5}$$

The reduction to a normalized aggregator $(f_2, 0)$ does not mean that intertemporal utility is risk neutral, or that the representation has lost the ability to separate risk aversion from substitution (see Duffie and Epstein (1992a)). The corresponding utility U_2 retains the essential features, namely that of (partly) disentangling intertemporal elasticity of substitution from risk aversion. This is the (standard) version analyzed previously by Duffie and Epstein (1992a) using dynamic programming.

The normalized version is used to prove existence and uniqueness of the solution to the backward stochastic differential equation (BSDE) (2), see Duffie and Epstein (1992b) and Duffie and Lions (1992).

Extending the analysis to more general stochastic processes, this is readily accomplished using the un-normalized aggregator (3), but has no counterpart in the representation (5). With jumps there will be at least one new term corresponding to A associated with risk aversion in connection with jump sizes, which can not be integrated in the manner of (5), without loosing the advantages of the extension: Recursive utility can be represented as

$$\begin{aligned} V_t = E_t\Big\{ \int_t^T \big(f(s,c_s,V_s) - \frac{1}{2}A(V_s)\, Z_s' Z_s \\ - \frac{1}{2}\int_{\mathcal{Z}} A_0(V_s,\zeta)K'(s,\zeta)K(s,\zeta)\nu(d\zeta)\big)ds\Big\}, \quad t \in [0,T] \end{aligned}$$

where $\nu(\cdot)$ is a Levy measure, $K(t,\cdot)$ is a square-integrable progressively measurable process related the jump part of the process, also to be determined by the associated BSDE (in addition to Z and V). The term $A_0(V_t,\cdot)$ penalizes for jump size risk, and is in general different from A (for details, see Aase (2015)). The model in this generality can be handled by the stochastic maximum principle.

However, for the model of this paper with diffusion driven uncertainty only, (3) and (5) have the same asset pricing implications. As long as the underlying stochastic processes are Markov, dynamic programing can be used. The stochastic maximum principle is more general.

A note on notation: The instantaneous correlation coefficient between, for example, returns and the consumption growth rate given by

$$\kappa_{Rc}(t) = \frac{\sigma_{Rc}(t)}{||\sigma_R(t)||\cdot||\sigma_c(t)||} = \frac{\sum_{i=1}^d \sigma_{R,i}(t)\sigma_{c,i}(t)}{\sqrt{\sum_{i=1}^d \sigma_{R,i}(t)^2}\sqrt{\sum_{i=1}^d \sigma_{c,i}(t)^2}},$$

when $d > 1$, and similarly for other correlations given in this model. Here $-1 \leq \kappa_{Rc}(t) \leq 1$ for all t. With this convention we can equally well write $\sigma_R'(t)\sigma_c(t)$ for $\sigma_{Rc}(t)$, and the former does *not* imply that the instantaneous correlation coefficient between returns and the consumption growth rate is equal to one.

2.1 Homogeneity

The following result will be made use of below. For a given consumption process c_t we let $(V_t^{(c)}, Z_t^{(c)})$ be the solution of the BSDE

$$\begin{cases} dV_t^{(c)} = \big(- f(t, c_t, V_t^{(c)}) + \frac{1}{2} A(V_t^{(c)})\, Z_t'^{(c)} Z_t^{(c)} \big)\, dt + Z_t'^{(c)}\, dB_t \\ V_T^{(c)} = 0 \end{cases} \tag{6}$$

Theorem 1 *Assume that, for all $\lambda > 0$,*
(i) $\lambda\, f(t,\ c,\ v) = f(t,\ \lambda c,\ \lambda v);\ \forall\ t,\ c,\ v,\ \omega$
(ii) $A(\lambda v) = \frac{1}{\lambda} A(v);\ \forall\ v$
Then

$$V_t^{(\lambda c)} = \lambda V_t^{(c)} \text{ and } Z_t^{(\lambda c)} = \lambda Z_t^{(c)},\ t \in [0, T]. \tag{7}$$

<u>Proof.</u> By uniqueness of the solution of the BSDEs of the type (6), all we need to do is to verify that the tuple $(\lambda V_t^{(c)}, \lambda Z_t^{(c)})$ is a solution of the BSDE (6) with c_t replaced by λc_t, i.e. that

$$\begin{cases} d(\lambda V_t^{(c)}) = \Big(- f(t, \lambda c_t, \lambda V_t^{(c)}) + \frac{1}{2} A(\lambda V_t^{(c)})\, \lambda Z_t'^{(c)} \lambda Z_t^{(c)} \Big)\, dt \\ +\lambda Z_t'^{(c)}\, dB_t; \qquad 0 \leq t \leq T \\ \lambda V_T^{(c)} = 0 \end{cases} \tag{8}$$

By (i) and (ii) and the quadratic variation interpretation of $Z'Z$, the BSDE (8) can be written

$$\begin{cases} \lambda dV_t^{(c)} = \Big(- \lambda f(t, c_t, V_t^{(c)}) + \frac{1}{2}\frac{1}{\lambda} A(V_t^{(c)})\, \lambda^2 Z_t'^{(c)} Z_t^{(c)} \Big)\, dt \\ +\lambda Z_t'^{(c)}\, dB_t; \qquad 0 \leq t \leq T \\ \lambda V_T^{(c)} = 0 \end{cases} \tag{9}$$

But this is exactly the equation (6) multiplied by the constant λ. Hence (9) holds and the proof is complete. $\square$

<u>Remarks</u> 1) Note that the system need not be Markovian in general, since we allow

$$f(t, c, v, \omega);\ (t, \omega) \in [0, T] \times \Omega$$

to be an adapted process, for each fixed c, v.
2) Similarly, we can allow A to depend on t as well [6].

Corollary 1 *Define $U(c) = V_0^{(c)}$. Then $U(\lambda c) = \lambda U(c)$ for all $\lambda > 0$.*

Notice that the aggregator in (3) satisfies the assumptions of the theorem.

3 The First Order Conditions

In the following we solve the consumer's optimization problem. The consumer is characterized by a utility function U and an endowment process e. For any of the versions $i = 1, 2$ formulated in the previous section, the representative agent's problem is to solve

$$\sup_{c \in L} U(c)$$

subject to

$$E\Big\{ \int_0^T c_t \pi_t dt \Big\} \leq E\Big\{ \int_0^T e_t \pi_t dt \Big\},$$

where π_t is the state price deflator. It represents the Arrow-Debreu state prices in units of probability, and plays a major role in this paper. Here $V_t = V_t^{(c)}$ and (V_t, Z_t) is the solution of the backward stochastic differential equation (BSDE)

$$\begin{cases} dV_t = -\tilde{f}(t, c_t, V_t, Z_t)\, dt + Z_t\, dB_t \\ V_T = 0. \end{cases} \tag{10}$$

Notice that (10) covers both the versions (3) and (5), where

$$\tilde{f}(t, c_t, V_t, Z_t) = f_i(c_t, V_t) - \frac{1}{2} A_i(V_t)\, Z_t' Z_t, \quad i = 1, 2.$$

Existence and uniqueness of solutions of the BSDE is treated in the general literature on this subject. For a reference see Theorem 2.5 in Øksendal and Sulem (2013), or Hu and Peng (1995). For the equation (10) existence and uniqueness follows from Duffie and Lions (1992).

For $\alpha > 0$ we define the Lagrangian

$$\mathcal{L}(c; \alpha) = U(c) - \alpha\, E\Big(\int_0^T \pi_t (c_t - e_t) dt \Big).$$

[6] although not standard in Economics.

Important is here that the quantity Z_t is part of the solution of the BSDE. Later we show how market clearing will finally determine the corresponding quantity in the market portfolio as a function of Z and the volatility σ_c of the growth rate of aggregate consumption. This internalizes prices in equilibrium.

In order to find the first order condition for the representative consumer's problem, we use Kuhn-Tucker and either directional (Frechet) derivatives in function space, or the stochastic maximum principle. Neither of these principles require any Markovian structure of the economy. The problem is well posed since U is increasing and concave and the constraint is convex. In maximizing the Lagrangian of the problem, we can calculate the directional derivative $\bigtriangledown U(c;h)$, alternatively denoted by$(\bigtriangledown U(c))(h)$, where $\bigtriangledown U(c)$ is the gradient of U at c. Since U is continuously differentiable, this gradient is a linear and continuous functional, and thus, by the Riesz representation theorem, it is given by an inner product. This we return to in Section 5.3.

Because of the generality of the problem, let us here utilize the stochastic maximum principle (see Pontryagin (1972), Bismut (1978), Kushner (1972), Bensoussan (1983), Øksendal and Sulem (2013), Hu and Peng (1995), or Peng (1990)): We then have a forward/backward stochastic differential equation (FBSDE) system consisting of the simple FSDE $dX(t) = 0; X(0) = 0$ and the BSDE (10)[7]. The Hamiltonian for this problem is

$$H(t, c, v, z, y) = y_t \, \tilde{f}(t, c_t, v_t, z_t) - \alpha \, \pi_t(c_t - e_t), \tag{11}$$

where y_t is the adjoint variable. Sufficient conditions for an optimal solution to the stochastic maximum principle can be found in the literature, see e.g., Theorem 3.1 in Øksendal and Sulem (2013). Hu and Peng (1995) also study existence and uniqueness of the solution to coupled FBSDE. A unique solution exists in the present case provided there is a unique solution to the BSDE (10); again Duffie and Lions (1992) is the appropriate reference.

The adjoint equation is

$$\begin{cases} dY_t = Y_t\big(\frac{\partial \tilde{f}}{\partial v}(t, c_t, V_t, Z_t)\, dt + \frac{\partial \tilde{f}}{\partial z}(t, c_t, V_t, Z_t)\, dB_t\big) \\ Y_0 = 1. \end{cases} \tag{12}$$

[7]The process X is part of the general formulation, not needed here and must be set equal to zero.

If c^* is optimal we therefore have

$$Y_t = \exp\Big(\int_0^t \{ \frac{\partial \tilde{f}}{\partial v}(s, c_s^*, V_s, Z_s) - \frac{1}{2}\big(\frac{\partial \tilde{f}}{\partial z}(s, c_s^*, V_s, Z_s)\big)^2 \} \, ds + \int_0^t \frac{\partial \tilde{f}}{\partial z}(s, c_s^*, V_s, Z_s) \, dB(s) \Big) \quad a.s. \quad (13)$$

Maximizing the Hamiltonian with respect to c gives the first order equation

$$y \, \frac{\partial \tilde{f}}{\partial c}(t, c^*, v, z) - \alpha \, \pi = 0$$

or

$$\alpha \, \pi_t = Y(t) \frac{\partial \tilde{f}}{\partial c}(t, c_t^*, V(t), Z_t) \quad \text{a.s. for all } t \in [0, T]. \quad (14)$$

Notice that the state price deflator π_t at time t depends, through the adjoint variable Y_t, on the entire optimal paths (c_s, V_s, Z_s) for $0 \leq s \leq t$. (The economy may be allowed to be non-Markovian since $\tilde{f}(\cdot)$ may also be allowed to depend on the state of nature.)

When $\gamma = \rho$ then $Y_t = e^{-\delta t}$ for the aggregator of the conventional model, which can be expressed as $f(c, v) = u(c) - \delta v$, $A = 0$. Thus the state price deflator is a Markov process, the utility is additive and dynamic programming can be used.

For the representative agent equilibrium the optimal consumption process is the given aggregate consumption c in society, and for this consumption process the utility V_t at time t is optimal.

We now have the first order conditions for both the versions of recursive utility outlined in Section 3. We analyze the non-ordinal version, denoted Model 1, with aggregator given by (3). The ordinally equivalent version (5) is analyzed in the Appendix.

4 The financial market

Having established the general recursive utility of interest, in this section we specify our model for the financial market. The model is much like the one used by Duffie and Epstein (1992a), except that we do not assume any unspecified factors in our model.

Let $\nu(t) \in \mathbb{R}^N$ denote the vector of expected rates of return of the N given risky securities in excess of the risk-less instantaneous return r_t, and let $\sigma(t)$ denote the matrix of diffusion coefficients of the risky asset prices, normalized by the asset prices, so that $\sigma(t)\sigma(t)'$ is the instantaneous covariance

matrix for asset returns. Both $\nu(t)$ and $\sigma(t)$ are progressively measurable, ergodic processes. For simplicity we assume that $N = d$, the dimension of the Brownian motion B.

The representative consumer's problem is, for each initial level w of wealth to solve

$$\sup_{(c,\varphi)} U(c) \tag{15}$$

subject to the intertemporal budget constraint

$$dW_t = \big(W_t(\varphi_t'\nu(t) + r_t) - c_t\big)dt + W_t\varphi_t'\sigma(t)dB_t. \tag{16}$$

Here $\varphi_t' = (\varphi_t^{(1)}, \varphi_t^{(2)}, \cdots, \varphi_t^{(N)})$ are the fractions of total wealth W_t held in the risky securities.

Market clearing requires that $\varphi_t'\sigma(t) = (\delta_t^M)'\sigma(t) = \sigma_M(t)$ in equilibrium, where $\sigma_M(t)$ is the volatility of the return on the market portfolio, and δ_t^M are the fractions of the different securities, $j = 1, \cdots, N$ held in the value-weighted market portfolio. That is, the representative agent must hold the market portfolio in equilibrium, by construction.

The model is a pure exchange economy where the aggregate endowment process e_t in society is exogenously given, and the single agent optimally consumes $c_t = e_t$ in every period, i.e., the agent optimally consumes the endowment process e_t at every date t. The main issue is then the determination of prices, including risk premiums and the interest rate, consistent with this behavior.

In the above we have interpreted the market portfolio as a proxy for the wealth portfolio, a common assumption in settings like this. This may, however be inaccurate. We return to this in Section 7.

5 The analysis of the non-ordinal model

We now turn our attention to pricing restrictions relative to the given optimal consumption plan. The first order conditions are

$$\alpha\,\pi_t = Y_t\,\frac{\partial f_1}{\partial c}(c_t, V_t) \quad \text{a.s. for all } t \in [0, T] \tag{17}$$

where f_1 is given in (3). The volatility Z_t and the utility process V_t satisfy the following dynamics

$$dV_t = \Big(-\frac{\delta}{1-\rho}\frac{c_t^{1-\rho} - V_t^{1-\rho}}{V_t^{-\rho}} + \frac{1}{2}\frac{\gamma}{V_t}Z'(t)Z_t\Big)dt + Z_t'dB_t \tag{18}$$

where $V(T) = 0$. This is the backward equation.

Aggregate consumption has the dynamics

$$\frac{dc_t}{c_t} = \mu_c(t)\,dt + \sigma_c(t)'\,dB_t, \tag{19}$$

where $\mu_c(t)$ and $\sigma_c(t)$ are measurable, $\mathcal{F}_t$ adapted stochastic processes, satisfying appropriate integrability properties. We assume these processes to be ergodic or stationary, so that we may 'replace' (estimate) time averages by state averages. In the Lucas (1978) model prices are determined in equilibrium such that the agent optimally consumes the endowment process e, which is exogenous to the analysis. This means that the process in (19) is eventually exogenous after market clearing in the 'fruit economy'.

The function $\tilde{f}$ of Section 4 is given by

$$\tilde{f}(t, c, v, z) = f_1(c, v) - \frac{1}{2}A(v)z'z,$$

and since $A(v) = \gamma/v$, from (12) the adjoint variable Y has dynamics

$$dY_t = Y_t\big(\{\frac{\partial}{\partial v}f_1(c_t, V_t) + \frac{1}{2}\frac{\gamma}{V_t^2}\,Z'(t)Z_t\}\,dt - A(V_t)Z_t'\,dB_t\big), \tag{20}$$

where $Y(0) = 1$. We now use the notation f for f_1 for simplicity. We also use the notation $Z_t/V_t = \sigma_V(t)$, valid for $V \neq 0$. By Theorem 1 the term $\sigma_V(t)$ is homogeneous of order zero in c.

We then seek the connection between $(V_t, \sigma_V(t))$ and the rest of the economy. Notice that Y is not a bounded variation process. From the FOC in (14) we obtain, by Ito's lemma, the dynamics of the state price deflator

$$d\pi_t = f_c(c_t, V_t)\,dY_t + Y_t\,df_c(c_t, V_t) + dY_t df_c(c_t, V_t), \tag{21}$$

where we have set $\alpha = 1$ without loss of generality. By the adjoint and the backward equations this is

$$\begin{aligned} d\pi_t &= Y_t\,f_c(c_t, V_t)\,\Big(\{f_v(c_t, V_t) + \frac{1}{2}\gamma\sigma_V'(t)\sigma_V(t)\}dt - \gamma\sigma_V(t)'dB_t\Big) \\ &+ Y_t\,\frac{\partial f_c}{\partial c}(c_t, V_t)\,dc_t + Y_t\,\frac{\partial f_c}{\partial v}(c_t, V_t)\,dV_t + dY_t df_c(c_t, V_t) \\ &+ Y_t\Big(\frac{1}{2}\,\frac{\partial^2 f_c}{\partial c^2}(c_t, V_t)\,(dc_t)^2 + \frac{\partial^2 f_c}{\partial c\,\partial v}(c_t, V_t)\,(dc_t)(dV_t) + \frac{1}{2}\,\frac{\partial^2 f_c}{\partial v^2}(c_t, V_t)\,(dV_t)^2\Big). \end{aligned} \tag{22}$$

Here

$$f_c(c, v) := \frac{\partial f(c, v)}{\partial c} = \delta c^{-\rho}v^{\rho},\ f_v(c, v) := \frac{\partial f(c, v)}{\partial v} = -\frac{\delta}{1-\rho}(1 - \rho c^{1-\rho}v^{\rho-1}),$$

$$\frac{\partial f_c(c,v)}{\partial c} = -\delta\rho c^{-(1+\rho)}v^{\rho}, \quad \frac{\partial f_c(c,v)}{\partial v} = \delta\rho v^{\rho-1}c^{-\rho},$$

$$\frac{\partial^2 f_c}{\partial c^2}(c,v) = \delta\rho(\rho+1)v^{\rho}c^{-(\rho+2)}, \quad \frac{\partial^2 f_c}{\partial c\,\partial v}(c,v) = -\delta\rho^2 v^{\rho-1}c^{-(\rho+1)},$$

and

$$\frac{\partial^2 f_c}{\partial v^2}(c,v) = \delta\rho(\rho-1)v^{\rho-2}c^{-\rho}.$$

5.1 The risk premiums

Denoting the dynamics of the state price deflator by

$$d\pi_t = \mu_\pi(t)\,dt + \sigma_\pi(t)'\,dB_t, \tag{23}$$

from (22) and the above expressions we obtain the drift and the diffusion terms of π_t as

$$\begin{aligned}\mu_\pi(t) = \pi_t\big(&-\delta - \rho\mu_c(t) + \frac{1}{2}\rho(\rho+1)\sigma_c'(t)\sigma_c(t) \\ &+ \rho(\gamma-\rho)\sigma_c'(t)\sigma_V(t) + \frac{1}{2}(\gamma-\rho)(1-\rho)\sigma_V'(t)\sigma_V(t)\big)\end{aligned} \tag{24}$$

and

$$\sigma_\pi(t) = -\pi_t\big(\rho\sigma_c(t) + (\gamma-\rho)\sigma_V(t)\big) \tag{25}$$

respectively.

Notice that $\mu_\pi(t)$ and $\sigma_\pi(t)$ depend on π_t, and the latter variable depends on consumption and utility from time zero to time t, since π_t depends on the adjoint variable Y_t, which is given by the expression

$$Y_t = \exp\Big(\int_0^t \big(\frac{\partial f}{\partial v}(s,c_s) + \frac{1}{2}\frac{\gamma(1-\gamma)}{V_s^2}Z'(s)Z_s\big)ds - \int_0^t \frac{\gamma}{V_s}Z_s dB_s\Big). \tag{26}$$

As can be seen, π_t depends on past consumption and utility from time zero to the present time t. Unless the terms $\mu_c(t)$, $\sigma_c(t)$, and $\sigma_V(t)$ are all deterministic, the state price process π_t is not a Markov process. If the parameters are deterministic in the conventional model, this implies that $\sigma_M(t) = \sigma_c(t)$ which is not supported by data (see Table 1 below). Hence these quantities must then be stochastic. If we allow this in the recursive model, our method still works, while dynamic programming is then ruled out - the stochastic maximum principle allows us to derive some optimality conditions without explicitly specifying the dependence.

Interpreting π_t as the price of the consumption good at time t, by the first order condition it is a decreasing function of consumption c since $f_{cc} < 0$.

The risk premium of any risky security with return process R is given by

$$\mu_R(t) - r_t = -\frac{1}{\pi_t}\,\sigma_\pi(t)'\,\sigma_R(t), \tag{27}$$

see e.g., Duffie (2001), Ch 10, eqn (37). It follows immediately from (25) and (27) that the formula for the risk premium of any risky security R is

$$\mu_R(t) - r_t = \rho\,\sigma_c(t)'\,\sigma_R(t) + (\gamma - \rho)\sigma_V(t)'\,\sigma_R(t). \tag{28}$$

This is our basic result for risk premiums.

It remains to connect $\sigma_V(t)$ to observables in the economy, which we do below. Before that we turn to the interest rate.

5.2 The equilibrium interest rate

The equilibrium short-term, real interest rate r_t is given by the formula

$$r_t = -\frac{\mu_\pi(t)}{\pi_t}. \tag{29}$$

The real interest rate at time t can be thought of as the expected exponential rate of decline of the representative agent's marginal utility, which is π_t in equilibrium (e.g., Duffie (2001)).

In order to find an expression for r_t in terms of the primitives of the model, we use (24). We then obtain the following

$$r_t = \delta + \rho\mu_c(t) - \frac{1}{2}\rho(\rho+1)\sigma_c(t)'\sigma_c(t) - \\ \rho(\gamma-\rho)\sigma_{cV}(t) - \frac{1}{2}(\gamma-\rho)(1-\rho)\sigma_V(t)'\sigma_V(t). \tag{30}$$

This is our basic result for the equilibrium short rate.

The potential for these two relationships to solve the puzzles should be apparent. We return to a discussion later.

We proceed to link the volatility term $\sigma_V(t)$ to observable quantities in the market that can be estimated from market data.

5.3 The determination of the volatility of the market portfolio.

In order to determine $\sigma_M(t)$ from the primitives $\sigma_V(t)$ and $\sigma_c(t)$, first notice that the wealth at any time t is given by

$$W_t = \frac{1}{\pi_t}E_t\Big(\int_t^T \pi_s c_s\,ds\Big), \tag{31}$$

where c is optimal. This expression follows since W can be interpreted as an asset that pays aggregate consumption as dividend. From Theorem 1 it follows that the non-ordinal utility function U is homogenous of degree one. Let c denote the stochastic process $\{c_t, 0 \leq t \leq T\}$. By the definition of directional derivatives we have that

$$\begin{aligned}\bigtriangledown U(c;c) &= \lim_{\alpha\downarrow 0}\frac{U(c+\alpha c)-U(c)}{\alpha} = \lim_{\alpha\downarrow 0}\frac{U(c(1+\alpha))-U(c)}{\alpha} \\ &= \lim_{\alpha\downarrow 0}\frac{(1+\alpha)U(c)-U(c)}{\alpha} = \lim_{\alpha\downarrow 0}\frac{\alpha U(c)}{\alpha} = U(c),\end{aligned}$$

where the third equality uses that U is homogeneous of degree one. By the Riesz representation theorem it follows from the linearity and continuity of the directional derivative that, by the first order condition

$$\bigtriangledown U(c;c) = E\Big(\int_0^T \pi_t c_t\, dt\Big) = W_0\pi_0 \tag{32}$$

where W_0 is the wealth of the representative agent at time zero, and the last equality follows from (31) for $t = 0$. Thus $U(c) = \pi_0 W_0$.

Let $V_t = V_t^{(c)}$ denote future utility at the optimal consumption for our representation. Since also V_t is homogeneous of degree one and continuously differentiable, by Riesz' representation theorem and the dominated convergence theorem, the same type of basic relationship holds here for the associated directional derivatives at any time t, i.e.,

$$\bigtriangledown V_t(c;c) = E_t\Big(\int_t^T \pi_s^{(t)}\, c_s\, ds\Big) = V_t(c)$$

where the Riesz representation $\pi_s^{(t)}$ for $s \geq t$ is the state price deflator at time $s \geq t$, as of time t. As for the discrete time model, it follows by results in Skiadas (2009a) that with assumption A2, implying that this quantity is independent of past consumption, the consumption history in the adjoint variable Y_t is 'removed' from the state price deflator π_t, so that $\pi_s^{(t)} = \pi_s/Y_t$ for all $t \leq s \leq T$. By this it follows that

$$V_t = \frac{1}{Y_t}\pi_t W_t. \tag{33}$$

This connects the dynamics of V to the rest of the economy. By the product rule,

$$dV_t = d\big(Y_t^{-1}\big)(\pi_t W_t) + Y_t^{-1} d(\pi_t W_t) + dY_t^{-1} d(\pi_t W_t). \tag{34}$$

where

$$d(\pi_t W_t) = W_t d\pi_t + \pi_t dW_t + d\pi_t dW_t \tag{35}$$

Ito's lemma gives

$$d\Big(\frac{1}{Y_t}\Big) = \Big(-\Big(\frac{1}{Y_t}\Big)\big(f_v(c_t, V_t) + \frac{1}{2}\gamma\sigma_V'(t)\sigma_V(t) + \frac{\gamma^2}{Y_t}\sigma_V'(t)\sigma_V(t)\Big)dt + \frac{1}{Y_t}\gamma\sigma_V(t)'dB_t \quad (36)$$

From the equations (34)-(36) it follows by the market clearing condition $\varphi_t' \cdot \sigma(t) = \sigma_M(t)$ that

$$V_t\sigma_V(t) = \frac{1}{Y_t}\Big(\pi_t W_t\gamma\sigma_V + \pi_t W_t\sigma_M(t) - \pi_t W_t\big(\rho\sigma_c(t) + (\gamma-\rho)\sigma_V(t)\big)\Big) \quad (37)$$

From the expression (33) for V_t we obtain the following equation for σ_V

$$\sigma_V(t) = \gamma\sigma_V(t) + \sigma_M(t) - (\rho\sigma_c(t) + (\gamma-\rho)\sigma_V(t)$$

from which it follows that

$$\sigma_M(t) = (1-\rho)\sigma_V(t) + \rho\sigma_c(t). \quad (38)$$

This is the internalization of M_t: The volatility of the market portfolio is a linear sum of the volatility of utility and the volatility of the growth rate of aggregate consumption, both parts of the primitives of the economic model. The quantities V and $\sigma_M(t)$ both exist as a solution to the BSDE (10). The pair (U, e) is given. In the Lucas model prices are determined in equilibrium such that the agent optimally consumes the endowment process e; hence the optimal consumption also becomes part of the primitives of the model.

This relationship can now be used to express $\sigma_V(t)$ in terms of the other two volatilities as

$$\sigma_V(t) = \frac{1}{1-\rho}(\sigma_M(t) - \rho\sigma_c(t)). \quad (39)$$

Alternatively, and somewhat easier, we can use the relation $V_tY_t = \pi_tW_t$ and the product rule directly to find these results.

Inserting the expression (39) into (28) and (30) we obtain the risk premiums

$$\mu_R(t) - r_t = \frac{\rho(1-\gamma)}{1-\rho}\,\sigma_c(t)'\sigma_R(t) + \frac{\gamma-\rho}{1-\rho}\,\sigma_M(t)'\sigma_R(t), \quad (40)$$

and the short rate

$$r_t = \delta + \rho\mu_c(t) - \frac{1}{2}\,\frac{\rho(1-\gamma\rho)}{1-\rho}\,\sigma_c(t)'\sigma_c(t) + \frac{1}{2}\,\frac{\rho-\gamma}{1-\rho}\,\sigma_M(t)'\sigma_M(t) \quad (41)$$

respectively.

The expression for the risk premium was derived by Duffie and Epstein (1992a) based on dynamic programming, assuming the volatilities involved to be constants. The expression for the real interest rate is new to this paper. The version treated by Duffie and Epstein (1992a) is the ordinally equivalent one based on (5).

The first covariance rate on the right-hand side of (40) is rather small, as in the conventional model, and can be ignored (for now). The second one is more significant. In order for this model to explain a large risk premium, consider, for example $\gamma > \rho$ and $\rho < 1$. The risk premium can be as large as one pleases by letting ρ be close enough to 1, for an otherwise reasonable value of γ. If $\gamma < \rho$ and $\rho > 1$, again the last term is positive, and can be arbitrarily large by letting ρ be close enough to 1, but notice that his combination may lead to a value of γ that is too low to be plausible.

Turning to the interest rate, in order for the model to explain a small short rate, consider the last term. When $\gamma > \rho$ and $\rho < 1$ this term is negative, and the variance of the wealth portfolio is larger than the corresponding variance rate of the consumption growth rate. The same argument again shows that this rate may be made as small as we please by letting ρ be close enough to 1. Not surprisingly, this gives us a reasonable fit, where also the parameter $\delta \geq 0$.

Also note that the effects from the last two terms in the short rate need not compensate for a very large term $\gamma\mu_c(t)$ in the conventional model, since this term is now $\rho\mu_c(t)$ with a reasonable value for ρ.

Weil (1989) on the other hand, had to his disposal only the following two expressions: The risk premium

$$-\frac{\mathrm{cov}_t(\mathcal{M}_{t+1}, R^R_{t+1})}{E_t(\mathcal{M}_{t+1})} = E_t(R^R_{t+1}) - R^f_{t+1}, \tag{42}$$

where the reciprocal of $E_t(\mathcal{M}_{t+1})$ is the gross rate of return R^f_{t+1} on the riskless asset over the period $(t, t+1)$. Here R^R_{t+1} is the gross return on a risky asset, while $\mathcal{M}_{t+1} = \frac{\pi_{t+1}}{\pi_t}$ is the stochastic discount factor (discrete time). Weil did not have the advantage to examine expressions like (40) and (41), and thus missed the interesting solution in the calibrations.

6 Summary of the model

Taking existence of equilibrium as given, the main results in this section are summarized as

Theorem 2 *For the non-ordinal model with aggregator $f_1(c, v)$, $A_1(v)$ specified in Sections 2-5, in equilibrium the risk premium of any risky asset is given by (40) and the real interest rate by (41).*

Using the same method for the version $(f_2(c, v), A_2(v))$, we obtain identical asset pricing implications, i.e., again (40) and (41) result. This is because monotonic transformations of utility functions do not affect the calculation of the marginal rate of substitution. The verification of this serves as a test of our methodology, i.e., the stochastic maximum principle works well for both versions.

Duffie and Epstein (1992a) derive the same risk premium using dynamic programming, but do not present an expression for equilibrium real interest rate.

The resulting risk premiums are linear combinations of the consumption-based CAPM and the market-based CAPM at each time t. The original derivation of the CAPM as an equilibrium model was given by Mossin (1966). His derivation was in a *time-less* setting, where the interest rate plays no role.

When the time preference $\rho = 0$ in Theorem 2, only the market-based CAPM remains. Accordingly, this model can be considered a *dynamic* version of the market-based CAPM, with the associated interest rate given by (41). In the present setting with recursive utility we denote this model by CAPM++. Below we also calibrate this version to the data summarized in Table 1 below. The last two terms in the short rate has, together with the expression for the equity premium, the potential to explain the low, observed values of the real rate, as we have seen. Also, when $\gamma > \rho$ the agent prefers early resolution of uncertainty to late (see Fig. 1).

The risk premium decreases as $\sigma_c(t)$ increases when $\gamma > \rho$ and $\rho < 1$. The conventional model can only predict an increase in the risk premium when this volatility increases. When $\sigma_M(t)$ increases in this situation, the interest rate decreases and the risk premium increases. The same happens if $\gamma < \rho$ and $\rho > 1$. The conventional model has no counterparts for this.

6.1 Calibrations

In Table 1 we present the key summary statistics of the data in Mehra and Prescott (1985), of the real annual return data related to the S&P-500, denoted by M, as well as for the annualized consumption data, denoted c, and the government bills, denoted b [8].

[8] There are of course newer data by now, but these retain the same basic features. If our model can explain the data in Table 1, it can explain any of the newer sets as well.

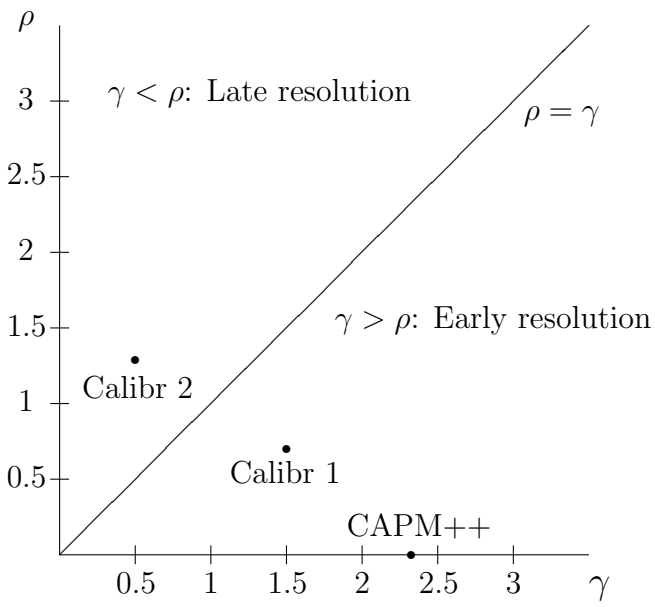

Figure 1: Calibration points in the (γ, ρ)-space

Here we have, for example, estimated the covariance between aggregate consumption and the stock index directly from the data set to be .00223. This gives the estimate .3770 for the correlation coefficient [9].

Since our development is in continuous time, we have carried out standard adjustments for continuous-time compounding, from discrete-time compounding. The results of these operations are presented in Table 2. This gives, e.g., the estimate $\hat{\kappa}_{Mc} = .4033$ for the instantaneous correlation coefficient $\kappa(t)$. The overall changes are in principle small, and do not influence our comparisons to any significant degree, but are still important.

First we interpret the risky asset R as the value weighted market portfolio M corresponding to the S&P-500 index. The conventional, additive Eu-model we obtain from (40) and (41) when $\gamma = \rho$. We then have two equations in two unknowns which provide estimates for the preference parameters by the "method of moments" [10]. The result for the Eu-model is $\gamma = 26.3$ and

[9] The full data set was provided by professor Rajnish Mehra.

[10] Implicitly this relies on an assumption about ergodicity/stationarity of the various μ_t and σ_t processes which enables us to replace "state averages" by "time averages", the

	Expectat.	Standard dev.	Covariances
Consumption growth	1.83%	3.57%	$\text{cov}(M, c) = .002226$
Return S&P-500	6.98%	16.54%	$\text{cov}(M, b) = .001401$
Government bills	0.80%	5.67%	$\text{cov}(c, b) = -.000158$
Equity premium	6.18%	16.67%	

Table 1: Key US-data for the time period 1889-1978. Discrete-time compounding.

	Expectation	Standard dev.	Covariances
Consumption growth	1.81%	3.55%	$\hat{\sigma}_{Mc} = .002268$
Return S&P-500	6.78%	15.84%	$\hat{\sigma}_{Mb} = .001477$
Government bills	0.80%	5.74%	$\hat{\sigma}_{cb} = -.000149$
Equity premium	5.98%	15.95%	

Table 2: Key US-data for the time period 1889-1978. Continuous-time compounding.

$\delta = -.015$, i.e., a relative risk aversion of about 26 and an impatience rate of minus 1.5%. This is the equity premium puzzle.

If we insist on a nonnegative impatience rate, as we probably should (but see Kocherlakota (1990)), this means that the real interest rate explained by the model is larger than 3.3% (when $\delta = .01$, say) for the period considered, but it is estimated, as is seen from Table 2, to be less than one per cent. The EIS parameter is calibrated to $\psi = .037$, which is considered to be too low for the representative individual.

There is of course some sampling error, so that these estimates are not exact, but clearly indicates that something is wrong with this model.

Calibrations of the model (40) and (41) are presented in Table 3, for plausible ranges of the parameters. We have consider Government bills as risk free[11].

As noticed, ρ can be constrained to be zero, in which case the model reduces to what we have called the CAPM++:

$$\mu_R(t) - r_t = \gamma\, \sigma_{M,R}(t), \qquad r_t = \delta - \frac{\gamma}{2}\sigma_M(t)'\sigma_M(t).$$

The risk premium is that of the ordinary CAPM-type, while the interest rate is new. This version of the model corresponds to "neutrality" of consumption

latter being given in Table 2.

[11]Calibrations sometimes give two different solutions, one close to the result from the conventional model, which is the one that Weil (1989) detected.

	γ	ρ	EIS	δ
The Expected utility model	26.37	26.37	.037	- .014
$\delta = .010$ fixed	.005	1.61	.62	.010
$\delta = .015$ fixed	.460	1.34	.74	.015
$\delta = .020$ fixed	.900	1.06	.94	.020
$\delta = .023$ fixed	1.15	.90	1.11	.023
$\delta = .030$ fixed	1.74	.48	2.08	.030
$\delta = .035$ fixed	2.14	.18	5.56	.035
$\rho = .90$ fixed	1.15	.90	1.11	.023
$\rho = .80$ fixed	1.30	.80	1.25	.025
$\rho = .50$ fixed	1.72	.50	2.00	.030
$\rho = .40$ fixed	1.86	.40	2.50	.031
CAPM ++	2.38	0.00	$+\infty$	.038
$\gamma = 0.50$ fixed	0.50	1.31	0.79	.015
$\gamma = 1.05$ fixed	1.05	.97	1.03	.040
$\gamma = 1.50$ fixed	1.50	.66	1.51	.027
$\gamma = 2.00$ fixed	2.00	.30	3.33	.033
$\gamma = 2.30$ fixed	2.30	.07	14.30	.036

Table 3: Calibrations Consistent with Table 2

transfers. Also, from the expression for the interest rate we notice that the short rate decreases in the presence of increasing market uncertainty. Solving these two non-linear equations consistent with the data of Table 2, we obtain

$$\gamma = 2.38 \quad \text{and} \quad \delta = .038.$$

In the conventional model this simply gives risk neutrality, i.e., $\gamma = \rho = 0$, so this model gives a risk premium of zero, and a short rate of $r = \delta$.

The original equilibrium model developed by Mossin (1966) was in a one period (a time-less) setting with consumption only on the terminal time point, in which case wealth equals consumption. Since there was no consumption at time 0, no intertemporal aspects of consumption transfers arose in the classical model. This naturally corresponds to $u(c) = c$ for the felicity index regarding consumption transfers, meaning $\rho = 0$ and $\psi = 1/\rho = +\infty$, and corresponding to perfect substitutability of consumption across time.

When the instantaneous correlation coefficient $\kappa_{Mc}(t)$ of returns and the aggregate consumption growth rate is small, our model handles this situation much better than the conventional one. The extreme case when $\kappa_{Mc}(t) = 0$ is, for example, consistent with the solution presented above for $\rho = 0$, which gives reasonable parameter values for the other parameters.

The main results in Bansal and Yaron (2004), mentioned in the introduction, are based on a risk aversion of 10 and an EIS of 1.5. In order to illustrate what a risk aversion of 10 might mean, consider a random variable X that takes the values 0 or 100 with probability 1/2 each. The equation $E\{u(100+X)\} := u(100+e_u)$ defines its certainty equivalent e_u at initial fortune 100 for the utility function u. If u is of power type $u(x) = \frac{x^{(1-\gamma)}}{1-\gamma}$, the certainty equivalent e_u is 50 when $\gamma = 0$, 33.3 when $\gamma = 2$, 7.98 when $\gamma = 10$, and 2.81 when $\gamma = 26$. Thus a risk aversion of 10 seems rather high.

Most of the plausible calibration points presented in Table 3 correspond to $\gamma > 1 > \rho$ and accordingly EIS > 1, for the data summarized in Table 2. Accordingly, these are located in the early resolution part of the (ρ, γ)-plane where $\gamma > \rho$. A value of EIS greater than 1 is consistent with the findings of Hansen and Singleton (1982) and many other authors (see below).

The present version is also consistent with calibrations in the region $0 < \gamma < \rho < 1$, corresponding to late resolution. As an example, if $\rho = 1.1$, this is consistent with $\delta = .02$ and $\gamma = .90$. The square root utility function is used in many examples in various textbooks (for the conventional model). For $\gamma = .5$ the model calibrates to $\delta = .015$ and $\rho = 1.31$, i.e., late resolution but otherwise for reasonable values of the parameters (calibration point Calibr 2 in Fig. 1). A value of $\gamma < 1$ seems less plausible, however.

6.2 Some new features of the model

It is reassuring that the risk premium of any risky asset depends on other investment opportunities in the financial market, and not just on this asset's covariance rate with consumption.

It is also satisfying that the return rate on government bonds depend on more than just the growth rate and the variance rate of aggregate consumption, but also on characteristics of other investment opportunities in the financial market.

Faced with increasing consumption uncertainty, the 'prudent' consumer will save and the interest rate accordingly falls in equilibrium (this is a fruit-tree economy). This is precautionary savings in the standard model. For recursive utility this property is more naturally linked to the last term in (41). When the wealth uncertainty increases, the interest rate falls provided $\gamma > \rho$ and $\rho < 1$, or $\gamma < \rho$ and $\rho > 1$. Furthermore, the equity premium increases in the same parameter ranges. As typical examples of the former, the calibration point Calibr 1 in Fig. 1 satisfies this requirement, as does the point CAPM++, while Calibr 2 satisfies the second.

This kind of discussion has no place in the conventional model, since when $\rho = \gamma$ there is no direct connection to the securities market (nor to the

wealth portfolio) in the expression for the risk premium (40). Similarly, the interest rate has no connection to the wealth portfolio in the conventional model, unlike for the recursive model.

The discrete-time recursive model of Epstein and Zin (1989-91) is the one that has mostly been used in applications. In evaluating the equity premium and the risk-free rate, some approximations must be carried out. In Aase (2013) the discrete-time model is solved using non-Markovian methods, and the pricing kernel turns out to be the same as the one obtained by Epstein and Zin (1989), and calibrated by Weil (1989). Unlike Weil, who used the same underlying two-state Markov model as fitted by Mehra and Prescott (1985), and numerically computed (42) and the gross short rate R^f_{t+1}, we work with testable and explicit expressions for risk premiums and the short rate. In discrete time our results are comparable to the ones of this paper (as the case should be).

6.3 Government bills

In the above discussion we have interpreted Government bills as risk free. With this in mind, there is another problem with the conventional, additive Eu-model. From Table 2 we see that there is a negative correlation between Government bills and the consumption growth rate. Similarly there is a positive correlation between the return on S&P-500 and Government bills.

If we interpret Government bills as risk free, the former correlation should be zero for the CCAPM-model to be consistent. Since this correlation is not zero, then γ must be zero, which is inconsistent with the above (and the model).

The Government bills used by Mehra and Prescott (1985) have duration one month, and the data are yearly, in which case Government bills are not the same as Sovereign bonds with duration of one year. One month bills in a yearly context will then contain price risk 11 months each year, and hence the estimate of the real, risk free rate is, perhaps, strictly lower that 0.80%. Whatever the positive value of the risk premium is, the resulting value of γ is negative. With bills included, the conventional, Eu-model does not seem to have enough 'degrees of freedom' to match the data, since in this situation the model contains three basic relationships and only two 'free parameters'.

The recursive model does much better in this regard, and yields more plausible results as it has enough degrees of freedom for this problem.

Exactly what risk premium bills command we can here only stipulate. For a risk premium of .0040 for the bills we have a third equation, namely

$$\mu_b(t) - r_t = \frac{\rho(1-\gamma)}{1-\rho}\,\sigma_{c,b}(t) + \frac{\gamma-\rho}{1-\rho}\,\sigma_{M,b} \tag{43}$$

to solve together with the equations (40) and (41). With the covariance estimates provided in Table 2, we have three equations in three unknowns, giving the following values $\delta = 0.027$, $\gamma = 1.76$ and $\rho = 0.53$. This risk premium of the bills indicates that the estimate of the real rate is only .0040, which may be a bit low, but these results are far better than the conventional, additive Eu-model can provide[12].

This may have several important consequences. To mention just one, recall the controversy around the Stern report, in which an estimate of 1.4 per cent for the real rate is suggested. Stern (2007) set the impatience rate $\delta = 0.001$, and received critique for this as well. Based on the above analysis, the real rate could have been set close to zero for climate related projects, and still have good model and empirical support.

7 The market portfolio is not a proxy for the wealth portfolio

In the paper we have focused on comparing two models, assuming the market portfolio can be used as a proxy for the wealth portfolio. Suppose we can view exogenous income streams as dividends of some shadow asset, in which case our model is valid if the market portfolio is expanded to include the new asset. However, if the latter is not traded, then the return to the wealth portfolio is not readily observable or estimable from available data. Still we should be able to get a pretty good impression of how the two models compare, which we now attempt.

In the conventional model with constant coefficients the growth rate of the wealth portfolio has the same volatility as the growth rate of aggregate consumption. Taking this quantity as the lower bound for this volatility, we indicate how the models compare when the market portfolio can not be taken as a proxy for the wealth portfolio. Below we first set $\sigma_W(t) = .05$, and $\kappa_{W,R} = .70$. The model can be written

$$\mu_M(t) - r_t = \frac{\rho(1-\gamma)}{1-\rho}\,\sigma_c(t)'\sigma_M(t) + \frac{\gamma-\rho}{1-\rho}\,\sigma_W(t)'\sigma_M(t), \tag{44}$$

and

$$r_t = \delta + \rho\mu_c(t) - \frac{1}{2}\,\frac{\rho(1-\rho\gamma)}{1-\rho}\,\sigma_c(t)'\sigma_c(t) + \frac{1}{2}\,\frac{\rho-\gamma}{1-\rho}\,\sigma_W(t)'\sigma_W(t). \tag{45}$$

[12]While Bansal and Yaron (2004) are off by a factor of 10 in explaining the volatility of the short rate (their Table IV), our model gives an exact fit to the data of Table 2.

Parameters	γ	ρ	EIS	δ
Recursive model				
$\delta = 10^{-6}$	2.73	.91	1.10	10^{-6}
$\delta = .001$	3.30	.88	1.36	.001
$\delta = .010$	7.86	.49	2.04	.010
$\delta = .015$	9.92	.18	5.55	.015
$\rho = 0.00$ CAPM++	10.80	.00	$+\infty$	.017
$\rho = .90$	3.01	.90	1.11	.000
$\rho = .85$	3.85	.85	1.18	.002
$\rho = .80$	4.60	.80	1.25	.003
$\rho = .70$	5.89	.70	1.43	.006
$\gamma = 2.80$	2.80	.91	1.10	.000
$\gamma = 3.50$	3.50	.87	1.14	.001
$\gamma = 4.00$	4.00	.84	1.19	.002
$\gamma = 4.50$	4.50	.81	1.23	.003

Table 4: Calibrations of the model when $\sigma_W(t) = .05$, and $\kappa_{W,R} = .70$.

Here M stands for the market portfolio and W for the wealth portfolio, so that (44) is the equity premium.

The calibrations are given in Table 4. The results are in favor of low values of the impatience rate δ. Typical values of γ fall between 2.7 and 4.5. The CAPM++ results when $\rho = 0$, and is here consistent with a (too) high value of $\gamma = 10.8$, but with a reasonable impatience rate of 1.7 per cent.

This value of .05 for the volatility of the wealth portfolio may be somewhat low. A more reasonable one is likely to be somewhere in between $\sigma_c(t)$ and $\sigma_M(t)$, so we suggest $\sigma_W(t) = .10$. We stipulate the correlation coefficient $\kappa_{W,M} = .80$. Calibrations under these assumptions are given in Table 5. As can be seen from the table, there is now a wide range of plausible solutions.

The illustrations in this section give a fairly clear indication of how the model performs when the market portfolio is not a proxy for the wealth portfolio. Many additional examples could of course be given, and the model can be extended and moved in various directions, as indicated by the extant literature. However, the examples presented are fairly simple, and give a reasonable illustration of how the recursive model behaves. Compared to the conventional model the difference is dramatic.

In addressing puzzles, it is desirable to alter as few features of the original model as possible at the time, in order to discover what made the difference. It seems like the change from the conventional representation of additive and

Parameters	γ	ρ	EIS	δ
Recursive model				
$\delta = .015$	0.93	1.01	.99	.015
$\delta = .020$	2.11	0.74	1.35	.020
$\delta = .025$	3.26	0.44	2.27	.025
$\delta = .030$	4.37	0.11	9.09	.030
$\rho = .000$	4.72	0.00	$+\infty$	.032
$\rho = .200$	4.08	0.20	5.00	.029
$\rho = .300$	3.75	0.30	3.33	.027
$\rho = .500$	3.04	0.50	2.00	.024
$\rho = .800$	1.87	0.80	1.25	.019
$\rho = .900$	1.44	0.90	1.11	.017
$\rho = .950$	1.22	0.95	1.05	.016
$\gamma = 1.01$	1.01	.99	1.01	.015
$\gamma = 2.00$	2.00	.77	1.30	.020
$\gamma = 2.50$	2.50	.64	1.56	.022
$\gamma = 3.00$	3.00	.51	1.96	.024
$\gamma = 3.50$	3.50	.37	2.70	.026
$\gamma = 4.00$	4.00	.23	4.35	.028
$\gamma = 4.50$	4.50	.07	14.29	.030

Table 5: Calibrations of the model when $\sigma_W(t) = .10$, and $\kappa_{W,R} = .80$.

separable expected utility to recursive utility, is just what it takes.

8 An empirical example

In this section we present the results of the Norwegian economy. This is a relatively small, open economy in which the central statistical agent, Statistisk sentralbyrå, has provided us with the data needed, also related to the wealth portfolio (from 1985 to 2013). Table 6 contains the data corresponding to Table 2.

The estimates provided by Statistisk sentralbyrå (2014) are restricted to include capital that is measurable in units of account: (i) human capital; (ii) real capital; (iii) financial capital (including the Sovereign Pension Fund of Norway); (iv) natural resources. For the whole period 72-75 per cent of the national wealth can be attributed to human capital.

Based on further information from the statistical agent, e.g., population

	Expectat.	Standard dev.	Covariances
Consumption growth	1.794%	1.390%	$\text{cov}(M, c) = .00078684$
Return OBX	10.70%	32.025%	$\text{cov}(M, b) = .00180603$
Government bills	2.141%	3.618%	$\text{cov}(c, b) = 1.0873E\text{-}05$
Equity premium	8.559%	31.703%	

Table 6: Key Norwegian-data for the time period 1971-2014 in terms of continuous-time compounding.

growth, the following per-capita estimates related to the wealth portfolio (log terms) are derived: $\sigma_W = .01849$, $\mu_W = .0219$, $\sigma_{W,M} = .00142$, $\sigma_{W,c} = .000127$. Below only the first and the third estimate are needed. The data on the wealth portfolio naturally represents a challenge to collect, and is associated with a fair amount of uncertainty; the presented estimates still gives a good indication of the national wealth. This gives us the calibrations of Table 7.

Parameters	γ	ρ	EIS	δ
$\gamma = 0.50$	0.50	1.004	.997	.013
$\gamma = 1.50$	1.50	.996	1.004	.013
$\gamma = 2.00$	2.00	.992	1.008	.013
$\gamma = 2.50$	2.50	.989	1.011	.013
$\gamma = 3.00$	3.00	.985	1.015	.014
$\gamma = 3.50$	3.50	.981	1.019	.014
$\gamma = 4.00$	4.00	.977	1.023	.014
$\gamma = 5.00$	5.00	.969	1.032	.014

Table 7: Calibrations of the recursive model to the Norwegian economy.

As can be seen, the values of the impatience rate vary little, so we have chosen to use the relative risk aversion parameter γ as the variable on the left-hand side of the table. The parameter estimates are reasonable over most of the range shown. When $\gamma = 0$, $\delta = 0.0128$ and $\rho = 1.0075$. When $\gamma = 25$, $\delta = .019$ and $\rho = .77$. This indicates a time preference $\rho \in (0.8, 1.0)$ and an impatience rate $\delta \in (0.012, 0.020)$. Thus relatively large variations in γ are associated with relatively small variations in the other two parameters. In conclusion, this indicates that the average Norwegian is reasonably patient, has an EIS just above 1 and has a relative risk aversion within a reasonable range. This is in accordance with Dagsvik et.al. (2006), who estimate EIS to be between 1 and 1.5 for the Norwegian population[13].

[13]The expected utility model calibrates to $\delta = -0.776$ and $\gamma = \rho = 108.78$ for this data

9 Extensions

The recursive models analyzed in this paper has been extended to include jump dynamics (Aase (2015)), which may be of particular interest in modeling stock market movements. This approach allows for an additional parameter γ_0 for risk aversion related to jump size risk, which can be different from γ. Here the stochastic maximum principle was indispensable. A heterogeneous model in continuous-time derived by the same methods gives comparable results (Aase (2014)).

The recursive utility maximizer takes into account more than just the present when evaluation the joint probability distribution of future states in the economy. As a result, the typical consumer smoothens consumption more distinctly than the Eu-maximizer, accordingly invests more in good times, allowing for more consumption in bad times. This behavior goes a long way in explaining the puzzle, provided the agent prefers early resolution of uncertainty to late.

10 Conclusions

We have addressed the well-known empirical deficiencies of the conventional asset pricing model in financial and macro-economics. The continuous-time recursive model is shown to fit data much better than the conventional Eu-model. Our formal approach is to use the stochastic maximum principle and forward/backward stochastic differential equations. This method can handle state dependence, which can be important in dealing with recursive utility.

In equilibrium the stochastic process of the the wealth portfolio is determined from the stochastic processes of utility and the the growth rate of aggregate consumption. With this in place, the model calibrates to plausible values of the parameters under reasonable assumptions.

When the market portfolio is not a proxy for the wealth portfolio, our results are the most interesting. For the Norwegian economy we have all the relevant data for the period 1971-2014, and perhaps surprisingly, the recursive model fits these data really well.

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set. The assumption that the economy is closed is of course restrictive, since it imposes that consumption be equal to domestic output. When exports and imports balance, this could still be a reasonable assumption.

References

[1] Aase, K. K. (2015). *"Recursive utility and jump-diffusions."* Working paper no. 6, Department of Business, Norwegian School of Economics, Bergen, Norway.

[2] Aase, K. K. (2014). *"Heterogeneity and limited stock market participation."* Working paper no 5, Department of Business, Norwegian School of Economics, Bergen, Norway.

[3] Aase, K. K. (2013). *"Recursive utility and the equity premium puzzle: A discrete-time approach."* Working paper no 3, Department of Business, Norwegian School of Economics, Bergen, Norway.

[4] Ai, H. (2010). "Information Quality and Long-run Risk: Asset Pricing Implications." *The Journal of Finance*, 64, 4, 1333-1367.

[5] Andersen, S., G. W. Harrison, M. I. Lau, and E. E. Rutström (2008) "Eliciting risk and time preferences." *Econometrica* 76, 3, 583-618.

[6] Avramov, D., S. Cederburg, and S. Hore (2010). *"Cross-Sectional Asset Pricing Puzzles: An equilibrium perspective."* Working Paper, University of Maryland.

[7] Avramov, D. and S. Hore (2008). *"Momentum, information uncertainty, and leverage - An explanation based on recursive preferences."* Working Paper, University of Maryland.

[8] Attanasio, O. P. (1999). "Consumption." *Handbook of Macroeconomics* (Ed: J.B. Taylor and M. Woodford), V1, 741-812.

[9] Azeredo, F. (2007). "Essays on aggregate economics and finance." Doctoral dissertation, University of California, Santa Barbara.

[10] Barro, R. J. (2006). "Rare disasters and asset markets in the twentieth century." *Quarterly Journal of Economics* 121, 3, 867-901.

[11] Bansal, R., and A. Yaron (2004). "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles." *The Journal of Finance*, 109, 4, 1481-1509.

[12] Bensoussan, A. (1983). *"Lectures on stochastic control."* In: Lect. Notes in Math. vol 972, 1-62. Springer, Berlin.

[13] Bollerslev,T., R. F. Engle, and J. Wooldridge (1988). "A capital asset pricing model with time-varying covariances." *Journal of Political Economy* 96, 116-131.

[14] Benartzi, S., and R. H. Thaler (1995). "Myopic loss aversion and the equity premium puzzle." *Quarterly Journal of Economics* 110 (1), 73-92.

[15] Bismut, J.-M. (1978). "An introductory approach to duality in optimal stochastic control." *SIAM Rev.* 20 (1), 62-78.

[16] Breeden, D. (1979). "An intertemporal asset pricing model with stochastic consumption and investment opportunities." *Journal of Financial Economics* 7, 265-296.

[17] Brown, S., W. N. Goetzmann, and S. A. Ross (1995). "Survival." *Journal of Finance* 50(3), 853-873.

[18] Browning, M., L. P. Hansen, and J. J. Heckman (1999). "Micro Data and General Equilibrium Models." *Handbook of Macroeconomics* (Ed: J.B. Taylor and M. Woodford), V1, 543-633.

[19] Cai, Y., K.L. Judd, and T.S. Lontzek (2013). "The social cost of stochastic and irreversible climate change." NBER working paper 18704.

[20] Cai, Y., K.L. Judd, and T.S. Lontzek (2015). "The social cost of carbon with economic and climate risks." Working paper, arXiv preprint. arXiv: 1504.06909.

[21] Campbell, J. (1993). "Intertemporal Asset Pricing without Consumption Data." *American Economic Review* 83, 487-512.

[22] Campbell, J. (1996). "Understanding Risk and Return." *Journal of Political Economy* 104, 298-345.

[23] Campbell, J. Y., and J. H. Cochrane (1999). "By force of habit: A consumption-based explanation of aggregate stock market behavior." *Journal of Political Economy* 107 (2), 205-51.

[24] Campbell, J. Y., and R. J. Shiller (1988). "The dividend-price ratio and expectations of future dividends and discount factors." *Review of Financial Studies* 1, 195-227.

[25] Chew, S., and L. Epstein (1991). "Recursive utility under uncertainty." In *Equilibrium theory in infinite dimensional spaces*, ed. by M. Ali Khan and Nicholas C. Yannelis, pp 352-369. (Studies in Economic Theory). Springer-Verlag: Berlin, Heidelberg, New York, London, Paris, Tokyo, Hong Kong, Barcelona, Budapest.

[26] Coccrane, J. H. (2008). "Financial markets and the real economy." In R. Mehra, ed. *Handbook of Equity Risk Premium*. Elsevier, Amsterdam.

[27] Constantinides, G. M. (1990). "Habit formation: a resolution of the equity premium puzzle." *Journal of Political Economy* 98, 519-543.

[28] Constantinides, G. M., J. B. Donaldson, and R. Mehra (2001). "Junior can't borrow: A new perspective on the equity premium puzzle." *Quarterly Journal of Economics* 107, 269-296.

[29] Cox, J., J. Ingersoll, and S. Ross (1985). "An intertemporal general equilibrium model of asset prices." *Econometrica* 53, 363-384.

[30] Dagsvik, J. K., S. Strøm, and Z. Jia (2006). "Utility of income as a random function: Behavioral characterization and empirical evidence." *Mathematical Social Sciences* 51, 23-57.

[31] Duffie, D. (1986). "Stochastic equilibria: Existence, spanning number and the 'no expected financial gain from trade' hypothesis." *Econometrica* 54, 1161-1183.

[32] Duffie, D. (2001). *Dynamic Asset Pricing Theory*, 3. ed., Princeton University Press, Princeton and Oxford.

[33] Duffie, D. and C. Skiadas (1994). "Continuous-time security pricing. A utility gradient approach." *Journal of Mathematical Economics* 23, 107-131.

[34] Duffie, D., P.-Y. Geoffard, and C. Skiadas (1994). "Efficient and equilibrium allocations with stochastic differential utility." *Journal of Mathematical Economics* 23, 133- 146.

[35] Duffie, D. and L. Epstein (1992a). "Asset pricing with stochastic differential utility." *Review of Financial Studies* 5, 411-436.

[36] Duffie, D. and L. Epstein (1992b). "Stochastic differential utility." *Econometrica* 60, 353-394.

[37] Duffie, D., and P. -L. Lions (1992). "PDE solutions of stochastic differential utility." *Journal of Mathematical Economics* 21, 577-606.

[38] Epstein, L., and S. Zin (1989). "Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework." *Econometrica* 57, 937-69.

[39] Epstein, L., and S. Zin (1991). "Substitution, risk aversion, and the temporal behavior of consumption and asset returns: An empirical analysis." *Journal of Policital Economy* 99, 263-286.

[40] Eraker, B., and Ivan Shaliastovich (2009). "An Equilibrium Guide to Designing Affine Pricing Models." *Mathematical Finance*, 18 (4), 519-543.

[41] Ferson, W. E. (1983). "Expectations of real interest rates and aggregate consumption: Empirical tests." *Journal of Financial and Quantitative Analysis* 18, 477-97.

[42] Guvenen, F. (2009). "A parsimonious macroeconomic model for asset pricing." *Econometrica* 77, 6, 1711-1750.

[43] Grossman, S. J., A. Melino, and R. J. Schiller (1987). "Estimating the continuous-time consumption-based asset-pricing model." *Journal of Business and Economic Statistics* 5, 315-27.

[44] Hansen, L. P., and K. J. Singleton (1982). "Generalized instrumental variables estimation of nonlinear rational expectations models." *Econometrica* 50, 1269-1286.

[45] Hansen, L. P., and K. J. Singleton (1983). "Stochastic consumption, risk aversion, and the temporal behavior of asset returns." *Journal of Political Economy* 91, 249-65.

[46] Hansen, L. P. and J. Scheinkman (2009). "Long Term Risk: An Operator Approach." *Econometrica* 77, 1, 177-234.

[47] Hansen, L.P., J. C. Heaton, N. Lee, and N. Roussanov (2007). "Intertemporal Substitution and Risk Aversion." *Handbook of Econometrics* Ch. 61, V. 6A, 3967-4056.

[48] Hansen, L. P., J. C. Heaton, and N. Li (2008). "Consumption Strikes back? Measuring Long-Run Risk." *Journal of Political Economy* 116, 2, 260-302.

[49] Heaton, J., and D. J. Lucas (1997). "Market frictions, saving behavior and portfolio choice." *Macroeconomic Dynamics* 1, 76-101.

[50] Hore, S. (2008). *"Equilibrium Predictability and other Return Characteristics."* Working Paper, University of Iowa.

[51] Hu, Y. and S. Peng (1995). "Solution of forward-backward stochastic differential equations." *Probab. Theory Relat. Fields*, 103, 273-285.

[52] Johnsen, T. H., and J. Donaldson (1985). "The Structure of Intertemporal Preferences under Uncertainty and Time Consistent Plans." *Econometrica* 53, 6, 1451-59.

[53] Kocherlakota, N. R. (1990a). "On the 'discount' factor in growth economies." *Journal of Monetary Economics* 25, 43-47.

[54] Kocherlakota, N. R. (1990b). "Disentangling the coefficient of relative risk aversion from the elasticity of intertemporal substitution: An irrelevancy result." *The Journal of Finance* 45, 175-190.

[55] Kocherlakota, N. R. (1996). "The Equity Premium: It's still a Puzzle." *Journal of Economic Literature* 34, 42-71.

[56] Koopmans, T. C. (1960). "Stationary ordinal utility and impatience." *Economctrica* 28, 287-309.

[57] Kreps, D. (1988). *"Notes on the Theory of Choice."* Underground Classics in Economics. Westview Press, Boulder and London.

[58] Kreps, D. and E. Porteus (1978). "Temporal resolution of uncertainty and dynamic choice theory." *Econometrica* 46, 185-200.

[59] Kreps, D. and E. Porteus (1979). "Dynamic choice theory and dynamic programming." *Econometrica* 47, 91-100.

[60] Kushner, N. J. (1972). Necessary conditions for continuous parameter stochastic optimization problems. *SIAM J. Control Optim.* 10, 550-565.

[61] Kydland, F. E., and E. C. Prescott (1982). "Time to build and aggregate fluctuations." *Econometrica* 50, 1345-70.

[62] Lucas, R. (1978). "Asset prices in an exchange economy." *Econometrica* 46, 1429-1445.

[63] McGrattan, E. R., and E. C. Prescott (2003). "Average Debt and Equity Returns: Puzzling?" *The American Economic Review* 93, 2, 392-397.

[64] Mehra, R., and E. C. Prescott (1985). "The equity premium: A puzzle." *Journal of Monetary Economics* 22, 145-161.

[65] Mehra, R., and E. C. Prescott (2008). "The Equity Premium: ABC's." Chapter 1 of R. Mehra, ed. *Handbook of Equity Risk Premium*, 1-36, Elsevier, Amsterdam.

[66] Mehra, R., and J. Donaldson (2008). "Risk-Based Explanations of the Equity Premium." Chapter 2 of R. Mehra, ed. *Handbook of Equity Risk Premium* 37-99, Elsevier, Amsterdam.

[67] Mossin, J. (1966). "Equilibrium in a capital asset market." *Econometrica* 34; 768-783.

[68] Mossin, J. (1969). "A Note on Uncertainty and Preferences in a Temporal Context." *The American Economic Review* 59, 1, 172-174.

[69] Øksendal, B. and A. Sulem (2013). *"Risk minimization in financial markets modeled by Itô-Levy processes."* Working paper, CMA, Dept. of Mathematics, University of Oslo.

[70] Peng, S. (1990). "A general stochastic maximum principle for optimal control problems." *SIAM J. Control Optim.* 28 (4), 966-979.

[71] Pindyck, R. S., and N. Wang (2013). "The Economic and Policy Consequences of Catastrophes." *American Economic Journal: Economic Policy*, 5(4): 306-339.

[72] Pontryagin, L.S. (1959). "Optimal control processes." *Usp. Mat. Nauk.*

[73] Rietz, T. A. (1988). "The equity risk premium: A solution." *Journal of Monetary Economics* 22, 133-136.

[74] Siegel, J. J. (1992). "The real rate of interest from 1800-1900." *Journal of Monetary Economics* 29, 227-252.

[75] Skiadas, C. (2009a). *"Asset Pricing Theory."* Princeton University Press, Princeton and Oxford.

[76] Skiadas, C. (2009b). *"Scale-Invariant Asset Pricing Theory: A general Discrete Framework with Ambiguity-Aversion Applications."* Working Paper, Kellogg School of Management, Northwestern University.

[77] Statistisk sentralbyrå (2014, June 11th). "Indikatorer for bærekraftig utvikling: Nasjonalformuen og bærekraftig utvikling." (https: www.ssb.no/statistikkbanken).

[78] Stern, N. (2007). *"The Economics of Climate Change."* Cambridge University Press.

[79] Wachter, J. A. (2008). *"Can time-varying risk of rare disasters explain aggregate stock market volatility?"* Working Paper, The Wharton School, University of Pennsylvania.

[80] Weil. P. (1989). "The Equity Premium Puzzle and the Risk-Free rate Puzzle." *Journal of Monetary Economics* 24, 401-421.

[81] Weil. P. (1990). "Non-Expected Utility in Macroeconomics." *Quarterly Journal of Economics* 105, 29-42.

[82] Weil, P. (1992). *"Equilibrium asset prices with undiversifiable labor income risk."* NBER Working Papers 3975. National Bureau of Economic Research, Inc.

[83] Weitzman, M. L. (2007). "Subjective expectations and asset-return puzzles." *American Economic Review* 97, 4, 1102-1130.