Theoretical Foundations of Buffer Stock Saving

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Drawbacks of Numerical Solutions

A Black Box

- Can Construct Solution to Model Without Really Understanding It
- Hard Even To Be Sure Your Numerical Solution Is *Right*
- Little Intuition for How Results Might Change With
    - Calibration
    - Structure
- *Very* Hard To Teach!

I Am A *Big* Fan Of Numerical Methods

- Have Done A Good Deal Of Work With Them Myself
- But As A Result, Have Felt All These Drawbacks Keenly

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Foundations For Microeconomic Household's Problem With

- Uncertain Labor Income
- No Liquidity Constraints
- CRRA Utility
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Key Result

Restrictions On Parameter Values Such That

- Problem Defines A Contraction Mapping
  - $\Rightarrow \quad \exists$ A Unique Consumption Function $\mathrm{c}(m)$
- There Is A ‘Target’ Ratio Of Assets to Permanent Income
  - Requires A Key ‘Impatience’ Condition To Hold
  - Good News
    - Condition Is Weaker (Easier To Satisfy) Than Previous Papers Imposed

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Limit as horizon T goes to infinity of

$$
\begin{aligned}
\mathbf{a}_t &= \mathbf{m}_t - \mathbf{c}_t \\
\mathbf{b}_{t+1} &= \mathbf{a}_t \mathsf{R} \\
\mathbf{p}_{t+1} &= \mathbf{p}_t \underbrace{\Gamma \psi_{t+1}}_{\equiv \Gamma_{t+1}} \\
\mathbf{m}_{t+1} &= \mathbf{b}_{t+1} + \mathbf{p}_{t+1} \xi_{t+1},
\end{aligned} \tag{1}
$$

$$
\xi_{t+n} = \begin{cases} 0 & \text{with probability } \wp > 0 \\ \theta_{t+n}/\cancel{\wp} & \text{with probability } \cancel{\wp} \equiv (1 - \wp) \end{cases} \tag{2}
$$

- $u(\bullet) = \bullet^{1-\rho}/(1-\rho)$; $\mathbb{E}_t[\psi_{t+n}] = \mathbb{E}_t[\xi_{t+n}] = 1 \ \forall \ n > 0$; $\beta < 1, \rho > 1$

Surely This Problem Has Been Solved?

No.

- Can't Use Stokey et. al. theorems because CRRA utility
- Lit thru **?** Can't Handle Permanent Shocks
- Must Use Boyd's 'Weighted' Contraction Mapping Theorem
- Surprisingly Subtle

Fortunately, the Conclusions Are Simple!

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Benchmark: Perfect Foresight Model

Definitions:

Absolute Patience Factor	Þ	$=$	$(\mathsf{R}\beta)^{1/\rho}$
Return Patience Factor	Þ_{R}	$=$	$\text{Þ}/\mathsf{R}$
Perfect Foresight Growth Patience Factor	Þ_{Γ}	$=$	$\text{Þ}/\Gamma$

Name	Condition	Implication
(AIC) Absolute Impatience Condition	$\text{Þ} < 1$	$c \downarrow$ over time
(RIC) Return Impatience Condition	$\text{Þ}_{\mathsf{R}} < 1$	$c/a \downarrow$ over time
(PF-GIC) Growth Impatience Condition	$\text{Þ}_{\Gamma} < 1$	$c/\boldsymbol{p} \downarrow$ over time

When Does A Useful Limiting Solution Exist?

Finite Human Wealth (FHWC) condition:

$$\Gamma < \mathsf{R} \tag{3}$$

Return Impatience Condition:

$$\text{Þ}_{\mathsf{R}} < \mathsf{R} \tag{4}$$

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What If There Are Liquidity Constraints?

- FHWC is *not* necessary for solution to exist
- Other Key Condition For Useful Solution is
  'Perfect Foresight Finite Value of Autarky Condition (PF-FVAC)':

$$\beta\Gamma^{1-\rho} < 1 \tag{5}$$

- Without RIC, Constraints Are Irrelevant
  - Because Wealth Always Wants To Rise, So Constraint Never Binds

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Liquidity Constraints and Uncertainty

- Introduce permanent shocks to income
- Finite Value of Autarky Condition Becomes

$$\begin{aligned} \overbrace{\beta \underline{\underline{\Gamma}}^{1-\rho}}^{\equiv \underline{\underline{\daleth}}} &< 1 \\ \beta &< \underline{\underline{\Gamma}}^{\rho-1} \end{aligned} \tag{6}$$

Contraction Mapping Requirements

Finite Value of Autarky Condition: Same As In Liq Constr Problem!

$$\begin{aligned} \overbrace{\beta\underline{\underline{\Gamma}}^{1-\rho}}^{\equiv\underline{\underline{\beth}}} &< 1 \\ \beta &< \underline{\underline{\Gamma}}^{\rho-1} \end{aligned} \tag{7}$$

'Weak Return Impatience Condition' (WRIC)

$$0 \leq \quad \wp^{1/\rho}\text{Þ}_{\mathsf{R}} \quad < 1 \tag{8}$$

Requirement For Existence Of A Target

Definitions: 'Uncertainty-Adjusted' Growth:

$$\underline{\Gamma} = \Gamma \acute{\psi} \tag{9}$$

Adjusted Growth Patience Factor:

$$Þ_{\acute{\Gamma}} = Þ/\underline{\Gamma} \tag{10}$$

Growth Impatience Condition:

$$Þ_{\acute{\Gamma}} < 1 \tag{11}$$

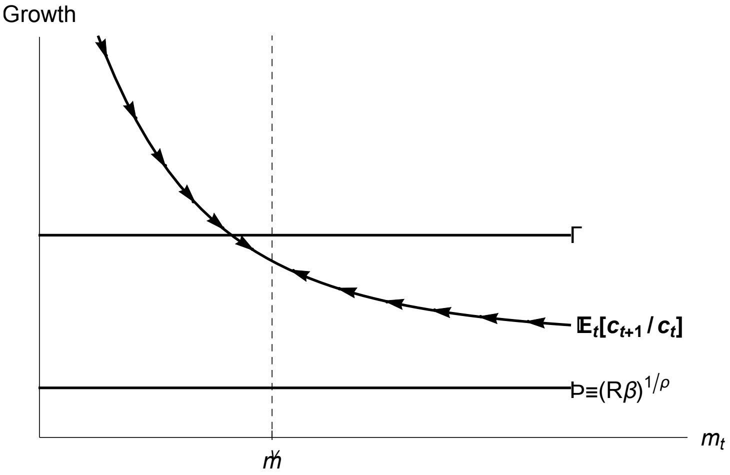
Why? Because it can be shown that

$$\lim_{m_t \to \infty} \mathbb{E}_t \left[\frac{m_{t+1}}{m_t} \right] = Þ_{\acute{\Gamma}} \tag{12}$$

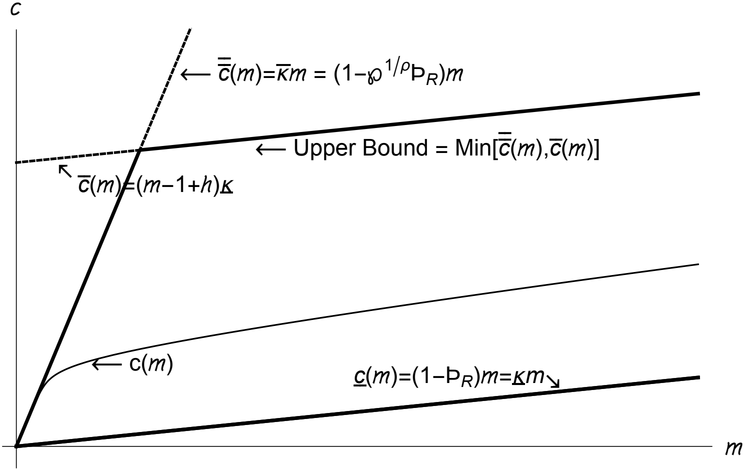
Five Propositions

1. $\lim_{m_t \to \infty} \mathbb{E}_t[c_{t+1}/c_t] = \text{Þ}$
2. $\lim_{m_t \to 0} \mathbb{E}_t[c_{t+1}/c_t] = \infty$
3. $\exists$ a unique target value of m, called $\check{m}$
4. $\mathbb{E}_t[c_{t+1}/c_t | m_t = \check{m}] = \Gamma - \epsilon$
5. $\left(\frac{d\mathbb{E}_t[c_{t+1}/c_t]}{dm_t} \right) < 0$

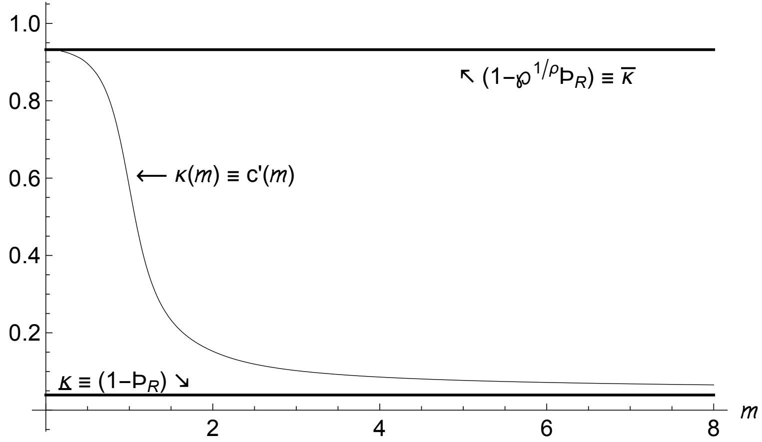
The Target Saving Figure



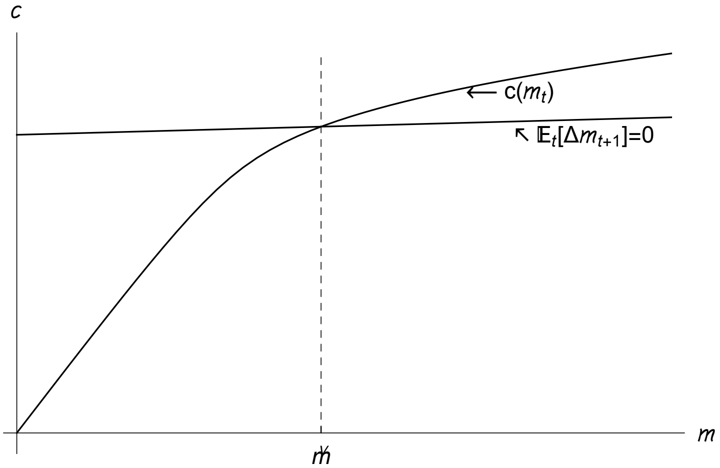
Bounds On the Consumption Function



The Marginal Propensity to Consume

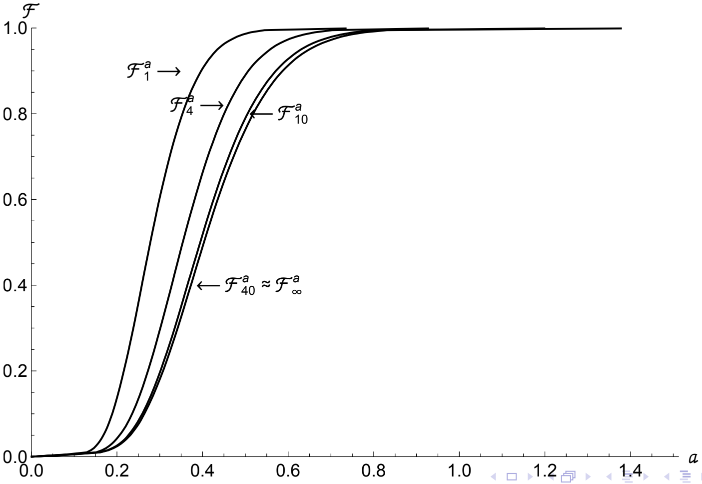


The Consumption Function and Target Wealth



Convergence To The Invariant Distribution

? Proves Existence of an Invariant Distribution of m, c, a, etc.



Balanced Growth Equilibrium

Achieved When Cross Section Distribution Reaches Invariance

$$Y_{t+1}/Y_t = C_{t+1}/C_t \quad = \quad \Gamma \tag{13}$$

Fisherian Separation Fails, Even Without Liquidity Constraints!

Insight:

- Precautionary Saving $\approx$ Liquidity Constraints
- If $\grave{\mathrm{c}}(m)$ is solution for constrained consumer,

$$\lim_{\wp \downarrow 0} \mathrm{c}(m; \wp) = \grave{\mathrm{c}}(m) \tag{14}$$

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The MPC Out Of Permanent Shocks

`http://econ.jhu.edu/people/ccarroll/papers/MPCPerm.pdf`

Lots of Recent Papers Trying to Measure the MPCP

Paper Proves:

- MPCP < 1
- But not a lot less:
  - 0.75 to 0.95 (annual rate) for wide range of parameter values

- Defined Conditions Under Which Widely Used Problem Has Solution
  - Finite Value of Autarky Condition Guarantees Contraction (with WRIC)
  - Growth Impatience Condition Prevents $m \rightarrow \infty$
- Economy Of Buffer Stock Consumers Exhibits Balanced Growth
  - Even In Absence of General Equilibrium Adj of Interest Rate

Matkowski, Janusz, and Andrzej S. Nowak (2011): "On Discounted Dynamic Programming With Unbounded Returns," *Economic Theory*, 46, 455–474.

Szeidl, Adam (2012): "Stable Invariant Distribution in Buffer-Stock Saving and Stochastic Growth Models," *Manuscript, Central European University*.