

# Online Supplemental Appendix to “Jump Factor Models in Large Cross-Sections”

Jia Li,<sup>\*</sup> Viktor Todorov,<sup>†</sup> and George Tauchen<sup>‡</sup>

November 7, 2018

## Abstract

This online supplement contains all proofs for the results in the main text.

---

<sup>\*</sup>Department of Economics, Duke University, Durham, NC 27708; e-mail: jl410@duke.edu.

<sup>†</sup>Department of Finance, Kellogg School of Management, Northwestern University, Evanston, IL 60208; e-mail: v-todorov@northwestern.edu.

<sup>‡</sup>Department of Economics, Duke University, Durham, NC 27708; e-mail: george.tauchen@duke.edu.

## SA Proofs

Throughout the proofs, we use  $K$  to denote a generic constant that may change from line to line. For a sub  $\sigma$ -field  $\mathcal{G} \subseteq \mathcal{F}$  and a sequence  $X_n$  of random variables, we write  $X_n \xrightarrow{\mathcal{L}|\mathcal{G}} X$  if the  $\mathcal{G}$ -conditional law of  $X_n$  converges in probability to that of  $X$  under a metric that is associated with the weak convergence of probability measures. By a standard localization procedure, we can strengthen Assumption 3 as the following without loss of generality:

**Assumption SA1.** *Suppose Assumption 3 holds with  $T_1 = \infty$ . Moreover, the processes  $\alpha_j$ ,  $\lambda_j$ ,  $\lambda_Z$ ,  $b_f$ ,  $\sigma_f$ ,  $\tilde{\sigma}_j^2$  and  $\tilde{J}_{Y,j}$  are bounded, uniformly in  $j$ .*

### SA.1 Preliminary Results

In this subsection, we introduce some notations and preliminary estimates that are used in the sequel. We consider a sequence  $\Omega_n$  of random events defined by  $\Omega_n \equiv \{\text{distinct jump times of the Poisson process } t \mapsto \mu([0, t], E) \text{ are at least } 2k_n\Delta_n \text{ apart}\}$ . Since  $k_n\Delta_n \rightarrow 0$  and the jumps of  $Z$  is of finite activity,  $\mathbb{P}(\Omega_n) \rightarrow 1$ . Therefore, we can restrict our calculations to  $\Omega_n$  without loss of generality. It is (notationally) convenient to extend the definition of the spot jump beta to all  $t \in [0, T]$  such that, on each path,  $\beta_{j,t} = \beta_{j,\tau}$  for  $t \in [\tau - k_n\Delta_n, \tau + k_n\Delta_n]$ . This extension is well-behaved on  $\Omega_n$  and our analysis only concerns the behavior of  $\beta_{j,t}$  around shrinking neighborhoods around the jump times. (It should be noted that  $\beta_{j,t}$  defined as such is not adapted to  $\mathcal{F}_t$ .)

We also consider the following sequence of events:

$$\Omega'_n \equiv \left\{ \sum_{j=1}^{N_n} 1_{\{\Delta_{i(n,\tau)}^n \tilde{J}_{Y,j} \neq 0\}} \leq \lfloor N_n q_n^w \rfloor / 2 \text{ for some } \tau \in \mathcal{T} \right\}. \quad (\text{SA.1})$$

By Markov's inequality,

$$\mathbb{P} \left( \sum_{j=1}^{N_n} 1_{\{\Delta_{i(n,\tau)}^n \tilde{J}_{Y,j} \neq 0\}} > \lfloor N_n q_n^w \rfloor / 2 \right) \leq \frac{2}{\lfloor N_n q_n^w \rfloor} \sum_{j=1}^{N_n} \mathbb{P} \left( \Delta_{i(n,\tau)}^n \tilde{J}_{Y,j} \neq 0 \right) \leq K \Delta_n / q_n^w \rightarrow 0.$$

Since  $\mathcal{T}$  is finite, we have  $\mathbb{P}(\Omega'_n) \rightarrow 1$ .

We denote the continuous part of  $Y_j$  and  $Z$  as, respectively,

$$Y'_{j,t} \equiv \int_0^t \alpha_{j,u} du + \int_0^t \lambda_{j,u}^\top df_u + \epsilon_{j,t}, \quad Z'_t \equiv \int_0^t \lambda_{Z,u}^\top df_u. \quad (\text{SA.2})$$

The diffusive residual process is then defined as

$$\tilde{Y}'_{j,t} \equiv Y'_{j,t} - \beta_{j,t} Z'_t = \int_0^t \alpha_{j,u} du + \left( \int_0^t \lambda_{j,u}^\top df_u - \beta_{j,t} \int_0^t \lambda_{Z,u}^\top df_u \right) + \epsilon_{j,t}. \quad (\text{SA.3})$$

We denote

$$\xi_{n,j,s} \equiv \frac{\Delta_{i(n,s)}^n \tilde{Y}_j'}{\Delta_{i(n,s)}^n Z}, \quad (\text{SA.4})$$

which can be decomposed as

$$\xi_{n,j,s} = \xi'_{n,j,s} + \xi''_{n,j,s}, \quad (\text{SA.5})$$

where

$$\left\{ \begin{array}{l} \xi'_{n,j,s} \equiv \frac{1}{\Delta Z_s} \left( \tilde{\lambda}_{j,s-}^\top (f_s - f_{(i(n,s)-1)\Delta_n}) + \tilde{\lambda}_{j,s}^\top (f_{i(n,s)\Delta_n} - f_s) + \epsilon_{i(n,s)\Delta_n} - \epsilon_{(i(n,s)-1)\Delta_n} \right), \\ \xi''_{n,j,s} \equiv \frac{1}{\Delta Z_s} \int_{(i(n,s)-1)\Delta_n}^{i(n,s)\Delta_n} \alpha_{j,u} du + \Delta_{i(n,s)}^n \tilde{Y}_j' \left( \frac{1}{\Delta_{i(n,s)}^n Z} - \frac{1}{\Delta Z_s} \right) \\ \quad + \frac{1}{\Delta Z_s} \int_{(i(n,s)-1)\Delta_n}^s (\lambda_{j,u} - \lambda_{j,s-})^\top df_u - \frac{\beta_{j,s}}{\Delta Z_s} \int_{(i(n,s)-1)\Delta_n}^s (\lambda_{Z,u} - \lambda_{Z,s-})^\top df_u \\ \quad + \frac{1}{\Delta Z_s} \int_s^{i(n,s)\Delta_n} (\lambda_{j,u} - \lambda_{j,s})^\top df_u - \frac{\beta_{j,s}}{\Delta Z_s} \int_s^{i(n,s)\Delta_n} (\lambda_{Z,u} - \lambda_{Z,s})^\top df_u. \end{array} \right. \quad (\text{SA.6})$$

We further rewrite  $\xi'_{n,j,s}$  as

$$\xi'_{n,j,s} = \Delta_n^{1/2} \sum_{q \in \{s-, s+\}} w_{n,q} \left( \tilde{\lambda}_{j,q}^\top \Sigma_{f,q}^{1/2} \zeta_{n,q} + R_{n,j,q} \right), \quad (\text{SA.7})$$

where we define

$$\left\{ \begin{array}{l} \zeta_{n,s-} \equiv \Sigma_{f,s-}^{-1/2} \frac{f_s - f_{(i(n,s)-1)\Delta_n}}{\sqrt{s - (i(n,s) - 1)\Delta_n}}, \quad \zeta_{n,s+} \equiv \Sigma_{f,s}^{-1/2} \frac{f_{i(n,s)\Delta_n} - f_s}{\sqrt{i(n,s)\Delta_n - s}}, \\ R_{n,j,s-} \equiv \frac{\epsilon_{j,s} - \epsilon_{j,(i(n,s)-1)\Delta_n}}{\sqrt{s - (i(n,s) - 1)\Delta_n}}, \quad R_{n,j,s+} \equiv \frac{\epsilon_{j,i(n,s)\Delta_n} - \epsilon_{j,s}}{\sqrt{i(n,s)\Delta_n - s}}, \\ w_{n,s-} \equiv \frac{1}{\Delta Z_s} \sqrt{\frac{s - (i(n,s) - 1)\Delta_n}{\Delta_n}}, \quad w_{n,s+} \equiv \frac{1}{\Delta Z_s} \sqrt{\frac{i(n,s)\Delta_n - s}{\Delta_n}}. \end{array} \right. \quad (\text{SA.8})$$

**Lemma SA1.** *Under Assumptions 3 and 4, we have for  $p, q \in \{\tau-, \tau+, \eta-, \eta+\}$ :*

- (a)  $N_n^{-1} \sum_{j=1}^{N_n} R_{n,j,p} R_{n,j,q} = O_p(N_n^{-1/2})$  when  $p \neq q$ ;
- (b)  $N_n^{-1} \sum_{j=1}^{N_n} R_{n,j,q}^2 \xrightarrow{\mathbb{P}} M_\epsilon(q)$ ;
- (c)  $N_n^{-1} \sum_{j=1}^{N_n} R_{n,j,p} \tilde{\lambda}_{j,q} = O_p(N_n^{-1/2})$ ;
- (d)  $N_n^{-1} \Delta_n^{-1} \sum_{j=1}^{N_n} (\xi''_{n,j,\tau} - \xi''_{n,j,\eta})^2 = O_p(\Delta_n)$ .

PROOF OF LEMMA SA1. (a) We prove the case with  $p = \tau-$  and  $q = \tau+$  in detail, while noting that the other cases can be proved in exactly the same way. Note that the jump times of the Poisson measure  $\mu$  are necessarily independent of the Brownian motions  $\tilde{W}_j$ ,  $1 \leq j \leq N_n$ . Let  $\mathcal{G}_t$  be the smallest filtration such that  $\mathcal{F}_t \subseteq \mathcal{G}_t$  and the jump times of  $\mu$  are  $\mathcal{G}_t$ -measurable. The processes  $(\tilde{W}_j)_{1 \leq j \leq N_n}$  remain to be Brownian motions with respect to  $(\mathcal{G}_t)_{t \geq 0}$ . Consequently,  $\epsilon_j$  is a  $(\mathcal{G}_t)_{t \geq 0}$ -martingale and, hence,

$$\mathbb{E}[R_{n,j,\tau-} R_{n,j,\tau+}] = 0. \quad (\text{SA.9})$$

Moreover, for  $j \neq m$ ,

$$\mathbb{E}[R_{n,j,\tau-}R_{n,j,\tau+}R_{n,m,\tau-}R_{n,m,\tau+}] = \mathbb{E}[R_{n,j,\tau-}R_{n,m,\tau-}\mathbb{E}[R_{n,j,\tau+}R_{n,m,\tau+}|\mathcal{G}_\tau]] = 0, \quad (\text{SA.10})$$

where the first equality holds because  $R_{n,j,\tau-}R_{n,m,\tau-}$  is  $\mathcal{G}_\tau$ -measurable and the second equality holds because  $\widetilde{W}_j$  and  $\widetilde{W}_m$  are orthogonal. Since the processes  $\tilde{\sigma}_j^2$  are uniformly bounded,  $\mathbb{E}[R_{n,j,\tau\pm}^4] \leq K$  holds due to a standard estimate for continuous Itô processes. By the Cauchy-Schwarz inequality, this further implies that

$$\mathbb{E}[R_{n,j,\tau-}^2R_{n,j,\tau+}^2] \leq K. \quad (\text{SA.11})$$

From (SA.9), (SA.10) and (SA.11), we deduce

$$\mathbb{E}\left[\left(N_n^{-1}\sum_{j=1}^{N_n}R_{n,j,p}R_{n,j,q}\right)^2\right] \leq KN_n^{-1}.$$

The assertion of part (a) then readily follows.

(b) We consider the case  $q = \tau-$  in details while noting that the other cases can be proved in the same way. By using Itô's formula, we can decompose

$$\begin{aligned} R_{n,j,\tau-}^2 &= U_{n,j} + U'_{n,j}, \quad \text{where} \\ U_{n,j} &\equiv \frac{1}{\tau - (i(n,\tau) - 1)\Delta_n} \int_{(i(n,\tau)-1)\Delta_n}^{\tau} \tilde{\sigma}_{j,u}^2 du, \\ U'_{n,j} &\equiv \frac{2}{\tau - (i(n,\tau) - 1)\Delta_n} \int_{(i(n,\tau)-1)\Delta_n}^{\tau} (\epsilon_{j,u} - \epsilon_{j,(i(n,\tau)-1)\Delta_n}) d\widetilde{W}_{j,u}. \end{aligned}$$

We note that  $\mathbb{E}[U'_{n,j}] = 0$  for each  $j$  and  $\mathbb{E}[U'_{n,j}U'_{n,m}] = 0$  for  $j \neq m$ . In addition,  $\mathbb{E}|U'_{n,j}|^2 \leq K$ . From these estimates, it readily follows that

$$\frac{1}{N_n} \sum_{j=1}^{N_n} U'_{n,j} = O_p(N_n^{-1/2}). \quad (\text{SA.12})$$

Next, note that by Assumption 3(v),

$$\mathbb{E}|U_{n,j} - \tilde{\sigma}_{j,\tau-}^2| \leq \mathbb{E}\left[\sup_{s,t,|s-t|\leq\Delta_n} |\tilde{\sigma}_{j,s}^2 - \tilde{\sigma}_{j,t}^2|\right] \leq K\Delta_n^{1/2}.$$

From this estimate and Assumption 4, we deduce

$$\frac{1}{N_n} \sum_{j=1}^{N_n} U_{n,j} = \frac{1}{N_n} \sum_{j=1}^{N_n} \tilde{\sigma}_{j,\tau-}^2 + o_p(1) = M_\epsilon(\tau-) + o_p(1). \quad (\text{SA.13})$$

The assertion of part (b) then follows from (SA.12) and (SA.13).

(c) By Assumption 4,  $\tilde{\lambda}_{j,q}$  is conditionally independent of  $R_{n,j,p}$  and, hence,  $\mathbb{E}[R_{n,j,p}\tilde{\lambda}_{j,q}|\mathcal{G}_0] = \mathbb{E}[R_{n,j,p}|\mathcal{G}_0]\mathbb{E}[\tilde{\lambda}_{j,q}|\mathcal{G}_0] = 0$ . In addition, for  $j \neq m$ ,

$$\mathbb{E}[R_{n,j,p}R_{n,m,p}\tilde{\lambda}_{j,q}\tilde{\lambda}_{m,q}^\top|\mathcal{G}_0] = \mathbb{E}[R_{n,j,p}R_{n,m,p}|\mathcal{G}_0]\mathbb{E}[\tilde{\lambda}_{j,q}\tilde{\lambda}_{m,q}^\top|\mathcal{G}_0] = 0,$$

where the second equality follows from the orthogonality between  $\tilde{W}_j$  and  $\tilde{W}_m$ . Since  $\tilde{\lambda}_{j,q}$  is bounded,  $R_{n,j,p}\tilde{\lambda}_{j,q}$  has bounded second moment. The assertion of part (c) readily follows from these facts.

(d) First, since the  $\alpha_j$ 's are uniformly bounded, it is easy to see that  $(\Delta Z_s)^{-1} \int_{(i(n,s)-1)\Delta_n}^{i(n,s)\Delta_n} \alpha_{j,u} du = O_p(\Delta_n)$  uniformly in  $j$ . Hence,

$$\frac{1}{N_n\Delta_n} \sum_{j=1}^{N_n} \left( \frac{1}{\Delta Z_s} \int_{(i(n,s)-1)\Delta_n}^{i(n,s)\Delta_n} \alpha_{j,u} du \right)^2 = O_p(\Delta_n). \quad (\text{SA.14})$$

Further note that, uniformly in  $j$ , we have  $\mathbb{E}|\Delta_{i(n,s)}^n \tilde{Y}'_j|^2 \leq K\Delta_n$  and, hence,

$$\frac{1}{N_n\Delta_n} \sum_{j=1}^{N_n} (\Delta_{i(n,s)}^n \tilde{Y}'_j)^2 = O_p(1). \quad (\text{SA.15})$$

It is easy to see that

$$\frac{1}{\Delta_{i(n,s)}^n Z} - \frac{1}{\Delta Z_s} = O_p(\Delta_n^{1/2}). \quad (\text{SA.16})$$

From (SA.15) and (SA.16), we deduce

$$\frac{1}{N_n\Delta_n} \sum_{j=1}^{N_n} \left( \Delta_{i(n,s)}^n \tilde{Y}'_j \left( \frac{1}{\Delta_{i(n,s)}^n Z} - \frac{1}{\Delta Z_s} \right) \right)^2 = O_p(\Delta_n). \quad (\text{SA.17})$$

We then note that, since the processes  $\lambda_j$ 's are (1/2)-Hölder continuous under  $L_2$ -norm uniformly in  $j$  (Assumption 3(v)), the following estimate also holds uniformly

$$\mathbb{E} \left[ \left( \int_{(i(n,s)-1)\Delta_n}^s (\lambda_{j,u} - \lambda_{j,s-})^\top df_u \right)^2 \right] \leq K\Delta_n^2.$$

Hence,

$$\frac{1}{N_n\Delta_n} \sum_{j=1}^{N_n} \left( \frac{1}{\Delta Z_s} \int_{(i(n,s)-1)\Delta_n}^s (\lambda_{j,u} - \lambda_{j,s-})^\top df_u \right)^2 = O_p(\Delta_n). \quad (\text{SA.18})$$

Similarly,

$$\left\{ \begin{array}{l} \frac{1}{N_n\Delta_n} \sum_{j=1}^{N_n} \left( \frac{\beta_{j,s}}{\Delta Z_s} \int_{(i(n,s)-1)\Delta_n}^s (\lambda_{Z,u} - \lambda_{Z,s-})^\top df_u \right)^2 = O_p(\Delta_n), \\ \frac{1}{N_n\Delta_n} \sum_{j=1}^{N_n} \left( \frac{1}{\Delta Z_s} \int_s^{i(n,s)\Delta_n} (\lambda_{j,u} - \lambda_{j,s})^\top df_u \right)^2 = O_p(\Delta_n), \\ \frac{1}{N_n\Delta_n} \sum_{j=1}^{N_n} \left( \frac{\beta_{j,s}}{\Delta Z_s} \int_s^{i(n,s)\Delta_n} (\lambda_{Z,u} - \lambda_{Z,s})^\top df_u \right)^2 = O_p(\Delta_n). \end{array} \right. \quad (\text{SA.19})$$

With an appeal to the Cauchy–Schwarz inequality, the assertion of part (d) then follows from (SA.14), (SA.17), (SA.18) and (SA.19). Q.E.D.

Next, we set

$$\begin{cases} A_n(s) \equiv \frac{1}{N_n \Delta_n} \sum_{j=1}^{N_n} (\xi'_{n,j,s})^2, & s \in \{\eta, \tau\}, \\ B_n(\eta, \tau) \equiv \frac{1}{N_n \Delta_n} \sum_{j=1}^{N_n} \xi'_{n,j,\eta} \xi'_{n,j,\tau}. \end{cases} \quad (\text{SA.20})$$

The following lemma collects some convergence results that we use for deriving limiting distributions.

**Lemma SA2.** *Suppose that Assumptions 3 and 4 hold. Then,*

$$(A_n(\eta), A_n(\tau), B_n(\eta, \tau)) \xrightarrow{\mathcal{L}\text{-s}} (\mathcal{A}(\eta), \mathcal{A}(\tau), \mathcal{B}(\eta, \tau)),$$

where  $\xrightarrow{\mathcal{L}\text{-s}}$  denotes  $\mathcal{F}$ -stable convergence in law.

PROOF OF LEMMA SA2. By Theorem 4.3.1 in Jacod and Protter (2012),

$$(w_{n,q}, \zeta_{n,q})_{q \in \{\eta^-, \eta^+, \tau^-, \tau^+\}} \xrightarrow{\mathcal{L}\text{-s}} (w_q, \zeta_q)_{q \in \{\eta^-, \eta^+, \tau^-, \tau^+\}}. \quad (\text{SA.21})$$

Recall the definitions in (SA.7) and (SA.20). We have, for  $s \in \{\eta, \tau\}$ ,

$$\begin{aligned} A_n(s) &= \frac{1}{N_n} \sum_{j=1}^{N_n} \left( \sum_{q \in \{s^-, s^+\}} w_{n,q} \left( \tilde{\lambda}_{j,q}^\top \Sigma_{f,q}^{1/2} \zeta_{n,q} + R_{n,j,q} \right) \right)^2 \\ &= \sum_{p,q \in \{s^-, s^+\}} w_{n,p} w_{n,q} \zeta_{n,p}^\top \Sigma_{f,p}^{1/2} \left( \frac{1}{N_n} \sum_{j=1}^{N_n} \tilde{\lambda}_{j,p} \tilde{\lambda}_{j,q}^\top \right) \Sigma_{f,q}^{1/2} \zeta_{n,q} \\ &\quad + \sum_{q \in \{s^-, s^+\}} w_{n,q}^2 \left( \frac{1}{N_n} \sum_{j=1}^{N_n} R_{n,j,q}^2 \right) + O_p(N_n^{-1/2}), \end{aligned} \quad (\text{SA.22})$$

where the rate for the  $O_p(N_n^{-1})$  term in the last line is obtained using Lemma SA1(a,c). Similarly,

$$\begin{aligned} B_n(\eta, \tau) &= \frac{1}{N_n} \sum_{j=1}^{N_n} \left( \sum_{p \in \{\tau^-, \tau^+\}} w_{n,p} \left( \tilde{\lambda}_{j,p}^\top \Sigma_{f,p}^{1/2} \zeta_{n,p} + R_{n,j,p} \right) \right) \\ &\quad \times \left( \sum_{q \in \{\eta^-, \eta^+\}} w_{n,q} \left( \tilde{\lambda}_{j,q}^\top \Sigma_{f,q}^{1/2} \zeta_{n,q} + R_{n,j,q} \right) \right) \\ &= \sum_{p \in \{\tau^-, \tau^+\}} \sum_{q \in \{\eta^-, \eta^+\}} w_{n,p} w_{n,q} \zeta_{n,p}^\top \Sigma_{f,p}^{1/2} \left( \frac{1}{N_n} \sum_{j=1}^{N_n} \tilde{\lambda}_{j,p} \tilde{\lambda}_{j,q}^\top \right) \Sigma_{f,q}^{1/2} \zeta_{n,q} + O_p(N_n^{-1/2}). \end{aligned} \quad (\text{SA.23})$$

By Assumption 4 and Lemma SA1(b),

$$\Sigma_{f,p}^{1/2} \left( \frac{1}{N_n} \sum_{j=1}^{N_n} \tilde{\lambda}_{j,p} \tilde{\lambda}_{j,q}^\top \right) \Sigma_{f,q}^{1/2} \xrightarrow{\mathbb{P}} M_C(p, q), \quad \frac{1}{N_n} \sum_{j=1}^{N_n} R_{n,j,q}^2 \xrightarrow{\mathbb{P}} M_\epsilon(q). \quad (\text{SA.24})$$

We further note that the limiting variables  $M_C(p, q)$  and  $M_\epsilon(q)$  are  $\mathcal{F}$ -measurable. Hence, by the property of stable convergence in law, we can deduce the assertion of Lemma SA2 from (SA.21), (SA.22), (SA.23) and (SA.24). *Q.E.D.*

Finally, we show in Lemma SA3 some consistency results for the spot jump beta estimates.

**Lemma SA3.** *Under Assumptions 3 and 5, the following holds for  $s \in \mathcal{T}$ :*

- (a)  $\sup_{1 \leq j \leq N_n} |\hat{\beta}_{n,j,s} - \beta_{j,s}| 1_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j} = 0\}} = o_p(1)$ ;
- (b)  $N_n^{-1} \sum_{j=1}^{N_n} |\hat{\beta}_{n,j,\tau} - \beta_{j,\tau}|^2 = o_p(1)$ .

PROOF OF LEMMA SA3. (a) Note that

$$\hat{\beta}_{n,j,s} - \beta_{j,s} = \frac{\Delta_{i(n,s)}^n Y_j - \beta_{j,s} \Delta_{i(n,s)}^n Z}{\Delta_{i(n,s)}^n Z}. \quad (\text{SA.25})$$

By localization, we can assume that  $\tilde{\sigma}_j^2$ ,  $\Sigma_f$ ,  $\lambda_j$  and  $\beta_j$  are bounded. By a standard estimate for continuous Itô semimartingales (applied to the continuous parts of  $Y_j$  and  $Z$ ), we have for any  $p \geq 1$ ,

$$\mathbb{E} \left[ \left| \Delta_{i(n,s)}^n Y_j - \beta_{j,s} \Delta_{i(n,s)}^n Z \right|^p 1_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j} = 0\}} \right] \leq K_p \Delta_n^{p/2},$$

for some constant  $K_p$ . By using a maximal inequality, we deduce that

$$\sup_{1 \leq j \leq N_n} \left| \Delta_{i(n,s)}^n Y_j - \beta_{j,s} \Delta_{i(n,s)}^n Z \right| 1_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j} = 0\}} = O_p(\Delta_n^{1/2} N_n^\iota) \quad (\text{SA.26})$$

for some arbitrarily small (but fixed) constant  $\iota > 0$ . Then, by Assumption 5,

$$\sup_{1 \leq j \leq N_n} \left| \Delta_{i(n,s)}^n Y_j - \beta_{j,s} \Delta_{i(n,s)}^n Z \right| 1_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j} = 0\}} = o_p(1).$$

Note that  $1/\Delta_{i(n,s)}^n Z = O_p(1)$ . The assertion of the lemma then readily follows from the above estimate and equation (SA.25).

(b) It is easy to see that  $\hat{\beta}_{n,j,\tau}$ ,  $1 \leq j \leq N_n$ , are uniformly bounded with probability approaching one. We then note that

$$\begin{aligned} & \frac{1}{N_n} \sum_{j=1}^{N_n} |\hat{\beta}_{n,j,\tau} - \beta_{j,\tau}|^2 \\ &= \frac{1}{N_n} \sum_{j=1}^{N_n} |\hat{\beta}_{n,j,\tau} - \beta_{j,\tau}|^2 1_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j} = 0\}} + \frac{1}{N_n} \sum_{j=1}^{N_n} |\hat{\beta}_{n,j,\tau} - \beta_{j,\tau}|^2 1_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j} \neq 0\}} \\ &\leq \left( \sup_{1 \leq j \leq N_n} |\hat{\beta}_{n,j,s} - \beta_{j,s}| 1_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j} = 0\}} \right)^2 + \frac{K}{N_n} \sum_{j=1}^{N_n} 1_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j} \neq 0\}} \\ &= o_p(1), \end{aligned}$$

as claimed in part (b).

*Q.E.D.*

## SA.2 Proof of Proposition 1

PROOF OF PROPOSITION 1. Recall that the spot jump betas  $\beta_{j,s}$  are bounded by assumption. By Lemma SA3 and the boundedness of  $\tilde{J}_{Y,j}$ , we further deduce that the beta estimates  $\hat{\beta}_{n,j,s}$  are uniformly (in  $j$ ) bounded with probability approaching one. Since the loss function  $L(\cdot)$  is Lipschitz on bounded sets (Assumption 1), we can now assume that  $L(\cdot)$  is globally Lipschitz without loss of generality. Hence, by Lemma SA3,

$$\begin{aligned}
& \frac{1}{N_n} \sum_{j=1}^{N_n} \left| L(\hat{\beta}_{n,j,\tau} - \hat{\beta}_{n,j,\eta}) - L(\chi_{j,\eta,\tau}) \right| \\
& \leq \frac{1}{N_n} \sum_{j=1}^{N_n} \left| L(\hat{\beta}_{n,j,\tau} - \hat{\beta}_{n,j,\eta}) - L(\chi_{j,\eta,\tau}) \right| \mathbf{1}_{\{|\Delta_{i(n,\tau)}^n \tilde{J}_{Y,j}| + |\Delta_{i(n,\eta)}^n \tilde{J}_{Y,j}| = 0\}} \\
& \quad + \frac{K}{N_n} \sum_{j=1}^{N_n} \mathbf{1}_{\{|\Delta_{i(n,\tau)}^n \tilde{J}_{Y,j}| + |\Delta_{i(n,\eta)}^n \tilde{J}_{Y,j}| > 0\}} \\
& \leq K \max_{s \in \{\eta, \tau\}, 1 \leq j \leq N_n} \left| \hat{\beta}_{n,j,s} - \beta_{j,s} \right| \mathbf{1}_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j} = 0\}} + O_p(\Delta_n) = o_p(1).
\end{aligned} \tag{SA.27}$$

Next, we set

$$\xi_n = \frac{1}{N_n} \sum_{j=1}^{N_n} (L(\chi_{j,\eta,\tau}) - \mathbb{E}[L(\chi_{j,\eta,\tau}) | \mathcal{F}_{\eta-}]).$$

Under Assumption 6,  $\xi_n$  is the average of  $\mathcal{F}_{\eta-}$ -conditionally independent variables with zero conditional mean. Hence,

$$\begin{aligned}
\mathbb{E} [\xi_n^2 | \mathcal{F}_{\eta-}] &= \frac{1}{N_n^2} \sum_{j=1}^{N_n} \mathbb{E} \left[ (L(\chi_{j,\eta,\tau}) - \mathbb{E}[L(\chi_{j,\eta,\tau}) | \mathcal{F}_{\eta-}])^2 \middle| \mathcal{F}_{\eta-} \right] \\
&\leq \frac{1}{N_n^2} \sum_{j=1}^{N_n} \mathbb{E} [L(\chi_{j,\eta,\tau})^2 | \mathcal{F}_{\eta-}] = O_p(N_n^{-1}) = o_p(1).
\end{aligned}$$

In particular, this implies that  $\mathbb{E} [|\xi_n| \wedge 1 | \mathcal{F}_{\eta-}] = o_p(1)$ . By the bounded convergence theorem, we further deduce  $\mathbb{E} [|\xi_n| \wedge 1] \rightarrow 0$ . But this is equivalent to  $\xi_n = o_p(1)$ . This, together with (SA.27), implies that

$$\frac{1}{N_n} \sum_{j=1}^{N_n} L(\hat{\beta}_{n,j,\tau} - \hat{\beta}_{n,j,\eta}) = \frac{1}{N_n} \sum_{j=1}^{N_n} \mathbb{E}[L(\chi_{j,\eta,\tau}) | \mathcal{F}_{\eta-}] + o_p(1). \tag{SA.28}$$

Since  $q_n^w \rightarrow 0$ , the winsorized estimator  $\hat{V}_n$  differs from  $N_n^{-1} \sum_{j=1}^{N_n} L(\hat{\beta}_{n,j,\tau} - \hat{\beta}_{n,j,\eta})$  by an  $o_p(1)$  term. The assertion of the proposition then follows from (SA.28). *Q.E.D.*



### SA.3 Proof of Theorem 1

PROOF OF THEOREM 1. Step 1. The proof proceeds in two steps. Recall  $\Omega'_n$  from (SA.1). Since  $\mathbb{P}(\Omega'_n) \rightarrow 1$ , we can restrict our calculations to  $\Omega'_n$  without loss of generality. In this step, we show that

$$\Delta_n^{-1} \hat{V}_n = \frac{1}{N_n \Delta_n} \sum_{j=1}^{N_n} L(\xi_{n,j,\tau} - \xi_{n,j,\eta}) + o_p(1). \quad (\text{SA.29})$$

From (SA.26), we see that

$$\sup_{1 \leq j \leq N_n} |\hat{\beta}_{n,j,s} - \beta_{j,s}| \mathbf{1}_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j} = 0\}} = O_p(\Delta_n^{1/2} N_n^\iota) \quad (\text{SA.30})$$

for some fixed but arbitrarily small constant  $\iota > 0$ . In restriction to  $\Omega'_n$  and the null hypothesis,  $\bar{B}_{n,\eta,\tau}$  is bounded by two times of the left-hand of the above display. Hence,

$$\bar{B}_{n,\eta,\tau} = O_p(\Delta_n^{1/2} N_n^\iota). \quad (\text{SA.31})$$

We note that

$$\begin{aligned} & \left| \frac{1}{\Delta_n N_n} \sum_{j=1}^{N_n} L \left( |\hat{\beta}_{n,j,\tau} - \hat{\beta}_{n,j,\eta}| \wedge \bar{B}_{n,\eta,\tau} \right) \right. \\ & \quad \left. - L \left( |\hat{\beta}_{n,j,\tau} - \hat{\beta}_{n,j,\eta}| \right) \mathbf{1}_{\{|\Delta_{i(n,\tau)}^n \tilde{J}_{Y,j}| + |\Delta_{i(n,\eta)}^n \tilde{J}_{Y,j}| = 0\}} \right| \\ & \leq \frac{\lceil q_n^w N_n \rceil}{\Delta_n N_n} \sup_{1 \leq j \leq N_n} L \left( |\hat{\beta}_{n,j,\tau} - \hat{\beta}_{n,j,\eta}| \right) \mathbf{1}_{\{|\Delta_{i(n,\tau)}^n \tilde{J}_{Y,j}| + |\Delta_{i(n,\eta)}^n \tilde{J}_{Y,j}| = 0\}} \\ & = O_p(q_n^w N_n^{2\iota}) = o_p(1), \end{aligned} \quad (\text{SA.32})$$

where the inequality follows from the fact that the winsorization is active for at most  $\lceil q_n^w N_n \rceil$  terms ( $\lceil \cdot \rceil$  denotes the ceiling function); the first equality follows from (SA.30); the second equality follows from Assumptions 2 and 5 with  $\iota$  chosen sufficiently small. Note that in restriction to  $\{|\Delta_{i(n,\tau)}^n \tilde{J}_{Y,j}| + |\Delta_{i(n,\eta)}^n \tilde{J}_{Y,j}| = 0\}$  and the null hypothesis,  $\hat{\beta}_{n,j,\tau} - \hat{\beta}_{n,j,\eta} = \xi_{n,j,\tau} - \xi_{n,j,\eta}$ . Hence,

$$\begin{aligned} & \frac{1}{\Delta_n N_n} \sum_{j=1}^{N_n} L \left( |\hat{\beta}_{n,j,\tau} - \hat{\beta}_{n,j,\eta}| \wedge \bar{B}_{n,\eta,\tau} \right) \mathbf{1}_{\{|\Delta_{i(n,\tau)}^n \tilde{J}_{Y,j}| + |\Delta_{i(n,\eta)}^n \tilde{J}_{Y,j}| = 0\}} \\ & = \frac{1}{\Delta_n N_n} \sum_{j=1}^{N_n} L(\xi_{n,j,\tau} - \xi_{n,j,\eta}) \mathbf{1}_{\{|\Delta_{i(n,\tau)}^n \tilde{J}_{Y,j}| + |\Delta_{i(n,\eta)}^n \tilde{J}_{Y,j}| = 0\}} + o_p(1). \end{aligned} \quad (\text{SA.33})$$

Next, we note that

$$\begin{aligned} & \frac{1}{N_n \Delta_n} \sum_{j=1}^{N_n} L \left( |\hat{\beta}_{n,j,\tau} - \hat{\beta}_{n,j,\eta}| \wedge \bar{B}_{n,\eta,\tau} \right) \mathbf{1}_{\{|\Delta_{i(n,\tau)}^n \tilde{J}_{Y,j}| + |\Delta_{i(n,\eta)}^n \tilde{J}_{Y,j}| > 0\}} \\ & \leq \frac{L(\bar{B}_{n,\eta,\tau})}{N_n \Delta_n} \sum_{j=1}^{N_n} \mathbf{1}_{\{|\Delta_{i(n,\tau)}^n \tilde{J}_{Y,j}| + |\Delta_{i(n,\eta)}^n \tilde{J}_{Y,j}| > 0\}} \\ & = O_p(\Delta_n N_n^{2\iota}) = o_p(1), \end{aligned} \quad (\text{SA.34})$$

where the inequality follows from the monotonicity of  $L(\cdot)$  and the last line follows from (SA.31) and the fact that  $\mathbb{P}(\Delta_{i(n,s)}^n \tilde{J}_{Y,j} \neq 0) \leq K\Delta_n$ . Similarly, we can show that

$$\frac{1}{N_n\Delta_n} \sum_{j=1}^{N_n} L(\xi_{n,j,\tau} - \xi_{n,j,\eta}) \mathbb{1}_{\{|\Delta_{i(n,\tau)}^n \tilde{J}_{Y,j}| + |\Delta_{i(n,\eta)}^n \tilde{J}_{Y,j}| > 0\}} = o_p(1). \quad (\text{SA.35})$$

From (SA.33), (SA.34) and (SA.35), we deduce (SA.29) as wanted.

Step 2. It remains to derive the convergence of  $(N_n\Delta_n)^{-1} \sum_{j=1}^{N_n} L(\xi_{n,j,\tau} - \xi_{n,j,\eta})$ . Recall the definition of  $\xi'_{n,j,s}$  from (SA.6). Let  $L_n$  be defined as

$$L_n \equiv \frac{1}{N_n\Delta_n} \sum_{j=1}^{N_n} (\xi'_{n,j,\tau} - \xi'_{n,j,\eta})^2. \quad (\text{SA.36})$$

Recalling the definitions in (SA.20), we can rewrite  $L_n$  as

$$L_n = A_n(\eta) + A_n(\tau) - 2B_n(\eta, \tau). \quad (\text{SA.37})$$

Then, by Lemma SA2,

$$L_n \xrightarrow{\mathcal{L}\text{-}s} \mathcal{L}(\eta, \tau) \equiv \mathcal{A}(\eta) + \mathcal{A}(\tau) - 2\mathcal{B}(\eta, \tau). \quad (\text{SA.38})$$

From (SA.5), we further see that

$$\begin{aligned} & \frac{1}{N_n\Delta_n} \sum_{j=1}^{N_n} L(\xi_{n,j,\tau} - \xi_{n,j,\eta}) \\ &= L_n + \frac{2}{N_n\Delta_n} \sum_{j=1}^{N_n} (\xi'_{n,j,\tau} - \xi'_{n,j,\eta})(\xi''_{n,j,\tau} - \xi''_{n,j,\eta}) + \frac{1}{N_n\Delta_n} \sum_{j=1}^{N_n} (\xi''_{n,j,\tau} - \xi''_{n,j,\eta})^2. \end{aligned} \quad (\text{SA.39})$$

By Lemma SA1(d), the last term in (SA.39) is  $O_p(\Delta_n)$ . By the Cauchy–Schwarz inequality, this estimate and (SA.38) further imply that the second term on the right-hand side of (SA.39) is  $o_p(1)$ .

Therefore,

$$\frac{1}{N_n\Delta_n} \sum_{j=1}^{N_n} L(\xi_{n,j,\tau} - \xi_{n,j,\eta}) = L_n + o_p(1).$$

The assertion of the theorem then follows from (SA.29) and (SA.38). *Q.E.D.*

## SA.4 Proof of Theorem 2

We start with the proof of part (a) and part (b). We provide details for the case with  $q = \tau-$ , while noting that the case with  $q = \tau+$  only requires a change of notation. Hence, we suppress (in most cases) the dependence on  $q$  in our notations for simplicity. More specifically, we write  $\hat{X}_n, \hat{F}_n, \hat{\Lambda}_n, \Lambda_n^*, \mathcal{E}_n, H, \Sigma_f, M_\Lambda^*$  and  $M_C^*$  in place of  $\hat{X}_n(q), \hat{F}_n(q), \hat{\Lambda}_n(q), \Lambda_n^*(q), \mathcal{E}_n(q), H_q, \Sigma_{f,q}, M_\Lambda^*(q, q)$  and  $M_C^*(q, q)$ , respectively. We denote the  $j$ th column of a generic matrix  $A$  by  $A_j$ .

Recall the sequence  $\Omega_n$  of events defined as in Section SA.1. Since  $\mathbb{P}(\Omega_n) \rightarrow 1$ , we can restrict our calculations below in  $\Omega_n$  without loss of generality.

Below, we denote  $\Gamma_n \equiv \{\gamma \in \mathbb{R}^{k_n} : \gamma^\top \gamma = k_n\}$ . Note that each column of  $\hat{F}_n$  is an element of  $\Gamma_n$ . We collect some useful estimates in Lemma SA4, where we denote

$$\tilde{\Lambda}_n^* \equiv \left( \lambda_{1,\tau-} - \tilde{\beta}_{n,1,\tau} \lambda_{Z,\tau-}, \dots, \lambda_{N_n,\tau-} - \tilde{\beta}_{n,N_n,\tau} \lambda_{Z,\tau-} \right)^\top. \quad (\text{SA.40})$$

We also consider an  $N_n \times k_n$  matrix  $\mathcal{E}'_n = [e'_{j,l}]_{1 \leq j \leq N_n, 1 \leq l \leq k_n}$  defined as

$$\begin{aligned} e'_{j,l} \equiv & \Delta_n^{-1/2} \int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n} \alpha_{j,s} ds + \Delta_n^{-1/2} \int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n} (\lambda_{j,u} - \lambda_{j,\tau-})^\top df_u \\ & - \tilde{\beta}_{n,j,\tau} \Delta_n^{-1/2} \int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n} (\lambda_{Z,u} - \lambda_{Z,\tau-})^\top df_u. \end{aligned} \quad (\text{SA.41})$$

**Lemma SA4.** *Under the conditions of Theorem 2, the following statements hold:*

- (a)  $\sup_{\gamma \in \Gamma_n} k_n^{-2} N_n^{-1} \gamma^\top \mathcal{E}_n^\top \mathcal{E}_n \gamma = o_p(1)$ ;
- (b)  $\sup_{\gamma \in \Gamma_n} k_n^{-2} N_n^{-1} \gamma^\top \mathcal{E}_n'^\top \mathcal{E}_n' \gamma = o_p(1)$ ;
- (c)  $\sup_{\gamma \in \Gamma_n} k_n^{-1} N_n^{-1} |\gamma^\top \mathcal{E}_n^\top \Lambda_n^*| = o_p(1)$ ;
- (d)  $\sup_{\gamma \in \Gamma_n} k_n^{-1} N_n^{-1} |\gamma^\top \mathcal{E}_n'^\top \Lambda_n^*| = o_p(1)$ ;
- (e)  $N_n^{-1} \tilde{\Lambda}_n^{*\top} \Lambda_n^* = M_\Lambda^* + o_p(1)$  and  $N_n^{-1} \tilde{\Lambda}_n^{*\top} \tilde{\Lambda}_n^* = M_\Lambda^* + o_p(1)$ .

PROOF OF LEMMA SA4. (a) Recall that the  $(j, l)$  element of  $\mathcal{E}_n$  is given by  $e_{j,l} \equiv \Delta_{i(n,\tau-)+l}^n \epsilon_j / \Delta_n^{1/2}$ .

We observe

$$\begin{aligned} & \frac{1}{k_n^2 N_n} \gamma^\top \mathcal{E}_n^\top \mathcal{E}_n \gamma \\ &= \frac{1}{k_n^2 N_n} \sum_{l=1}^{k_n} \sum_{m=1}^{k_n} \gamma_l \gamma_m \sum_{j=1}^{N_n} e_{j,l} e_{j,m} \\ &\leq \left( \frac{1}{k_n^2} \sum_{l=1}^{k_n} \sum_{m=1}^{k_n} \gamma_l^2 \gamma_m^2 \right)^{1/2} \left( \frac{1}{k_n^2} \sum_{l=1}^{k_n} \sum_{m=1}^{k_n} \left( \frac{1}{N_n} \sum_{j=1}^{N_n} e_{j,l} e_{j,m} \right)^2 \right)^{1/2} \\ &= \left( \frac{1}{k_n^2} \sum_{l=1}^{k_n} \sum_{m=1}^{k_n} \left( \frac{1}{N_n} \sum_{j=1}^{N_n} e_{j,l} e_{j,m} \right)^2 \right)^{1/2}, \end{aligned} \quad (\text{SA.42})$$

where the first equality is by definition, the inequality is by the Cauchy–Schwarz inequality, and the last line follows from  $\gamma^\top \gamma = k_n$ .

We decompose the majorant side of (SA.42) as

$$\begin{aligned} & \frac{1}{k_n^2} \sum_{l=1}^{k_n} \sum_{m=1}^{k_n} \left( \frac{1}{N_n} \sum_{j=1}^{N_n} e_{j,l} e_{j,m} \right)^2 \\ &= \frac{1}{k_n^2} \sum_{l=1}^{k_n} \left( \frac{1}{N_n} \sum_{j=1}^{N_n} e_{j,l}^2 \right) + \frac{1}{k_n^2} \sum_{l,m,l \neq m} \left( \frac{1}{N_n} \sum_{j=1}^{N_n} e_{j,l} e_{j,m} \right)^2. \end{aligned} \quad (\text{SA.43})$$

By a standard estimate for continuous Itô semimartingales,  $\mathbb{E}[e_{j,l}^4] \leq K$ ; this holds uniformly in  $j \in \{1, \dots, N_n\}$  because the idiosyncratic variances  $\tilde{\sigma}_j^2$  are uniformly (locally) bounded under Assumption 3(iii). Hence, by Jensen's inequality,

$$\mathbb{E} \left[ \left( \frac{1}{N_n} \sum_{j=1}^{N_n} e_{j,l}^2 \right)^2 \right] \leq \mathbb{E} \left[ \frac{1}{N_n} \sum_{j=1}^{N_n} e_{j,l}^4 \right] \leq K.$$

From here, it follows that the first term on the right-hand side of (SA.43) is  $o_p(1)$ . In view of (SA.42) and (SA.43), it remains to show that the second term on the right-hand side of (SA.43) is also  $o_p(1)$ .

To this end, we observe the following for  $l \neq m$ : (i)  $\mathbb{E}[e_{j,l}e_{j,m}] = 0$  because the process  $\epsilon_j$  is a martingale; (ii)  $\mathbb{E}[e_{j,l}^2 e_{j,m}^2] \leq K$ ; and (iii) the variables  $(e_{j,l}e_{j,m})_{1 \leq j \leq N_n}$  are uncorrelated, which can be shown by using repeated conditioning and the orthogonality among the Brownian motions  $(\widetilde{W}_j)_{j \geq 1}$ . Hence,

$$\mathbb{E} \left[ \left( \frac{1}{N_n} \sum_{j=1}^{N_n} e_{j,l}e_{j,m} \right)^2 \right] \leq K N_n^{-1} \rightarrow 0,$$

which implies, as wanted,

$$\frac{1}{k_n^2} \sum_{l,m,l \neq m} \left( \frac{1}{N_n} \sum_{j=1}^{N_n} e_{j,l}e_{j,m} \right)^2 = O_p(N_n^{-1}) = o_p(1).$$

This finishes the proof of part (a).

(b) Similar to (SA.42), we can derive

$$\sup_{\gamma \in \Gamma_n} \frac{1}{k_n^2 N_n} \gamma^\top \mathcal{E}'_n \mathcal{E}'_n \gamma \leq \left( \frac{1}{k_n^2} \sum_{l=1}^{k_n} \sum_{m=1}^{k_n} \left( \frac{1}{N_n} \sum_{j=1}^{N_n} e'_{j,l} e'_{j,m} \right)^2 \right)^{1/2}. \quad (\text{SA.44})$$

In addition, we observe

$$\begin{aligned} & \mathbb{E} \left| \Delta_n^{-1/2} \int_{(i(n,\tau^-)+l-1)\Delta_n}^{(i(n,\tau^-)+l)\Delta_n} (\lambda_{j,u} - \lambda_{j,\tau^-})^\top df_u \right|^4 \\ & \leq K \Delta_n^{-2} \mathbb{E} \left[ \left( \int_{(i(n,\tau^-)+l-1)\Delta_n}^{(i(n,\tau^-)+l)\Delta_n} \|\lambda_{j,u} - \lambda_{j,\tau^-}\|^2 du \right)^2 \right] \\ & \leq K \Delta_n^{-1} \mathbb{E} \left[ \int_{(i(n,\tau^-)+l-1)\Delta_n}^{(i(n,\tau^-)+l)\Delta_n} \|\lambda_{j,u} - \lambda_{j,\tau^-}\|^4 du \right] \\ & \leq K \Delta_n^{-1} \mathbb{E} \left[ \int_{(i(n,\tau^-)+l-1)\Delta_n}^{(i(n,\tau^-)+l)\Delta_n} \|\lambda_{j,u} - \lambda_{j,\tau^-}\|^2 du \right] \leq K \Delta_n, \end{aligned} \quad (\text{SA.45})$$

where the first inequality is by the Burkholder–Davis–Gundy inequality, the second inequality is by Jensen's inequality, and the last line holds because  $\lambda_{j,u}$  is bounded and (1/2)-Hölder continuous

under  $L_2$ -norm uniformly in  $j$ . Similarly,

$$\mathbb{E} \left| \Delta_n^{-1/2} \int_{(i(n,\tau^-)+l-1)\Delta_n}^{(i(n,\tau^-)+l)\Delta_n} (\lambda_{Z,u} - \lambda_{Z,\tau^-})^\top df_u \right|^4 \leq K \Delta_n. \quad (\text{SA.46})$$

Under Assumption 7,  $(\tilde{\beta}_{j,n,\tau})_{1 \leq j \leq N_n}$  are uniformly bounded with probability approaching one, so we can assume that these variables are bounded without loss of generality. Hence, from (SA.45) and (SA.46), we deduce that

$$\mathbb{E}|e'_{j,l}|^4 \leq K \Delta_n. \quad (\text{SA.47})$$

Hence, by the Cauchy–Schwarz inequality, we further have

$$\mathbb{E} \left[ \left( \frac{1}{N_n} \sum_{j=1}^{N_n} e'_{j,l} e'_{j,m} \right)^2 \right] \leq K \Delta_n. \quad (\text{SA.48})$$

The assertion of part (b) then follows from (SA.44) and (SA.48).

(c) We denote the  $(j, k)$  element of  $\Lambda_n^*$  by  $\lambda_{j,k}^*$ . We note that for each  $k \in \{1, \dots, r\}$  (recalling that  $\Lambda_{n,k}^*$  denotes the  $k$ th column of  $\Lambda_n^*$ ),

$$\begin{aligned} \frac{1}{k_n N_n} \left| \gamma^\top \mathcal{E}_n^\top \Lambda_{n,k}^* \right| &= \left| \frac{1}{k_n} \sum_{l=1}^{k_n} \gamma_l \left( \frac{1}{N_n} \sum_{j=1}^{N_n} e_{j,l} \lambda_{j,k}^* \right) \right| \\ &\leq \left( \frac{1}{k_n} \sum_{l=1}^{k_n} \gamma_l^2 \right)^{1/2} \left( \frac{1}{k_n} \sum_{l=1}^{k_n} \left( \frac{1}{N_n} \sum_{j=1}^{N_n} e_{j,l} \lambda_{j,k}^* \right)^2 \right)^{1/2} \\ &= \left( \frac{1}{k_n} \sum_{l=1}^{k_n} \left( \frac{1}{N_n} \sum_{j=1}^{N_n} e_{j,l} \lambda_{j,k}^* \right)^2 \right)^{1/2}, \end{aligned} \quad (\text{SA.49})$$

where the first line is by definition, the second line is by the Cauchy–Schwarz inequality and the last line follows from  $\gamma \in \Gamma_n$ . Under Assumption 8,  $e_{j,l}$  is independent of  $\lambda_{j,k}^*$ ; hence, the variables  $(e_{j,l} \lambda_{j,k}^*)_{1 \leq j \leq N_n}$  are uncorrelated and have zero mean and bounded second moment. It is then easy to see that

$$\mathbb{E} \left[ \frac{1}{k_n} \sum_{l=1}^{k_n} \left( \frac{1}{N_n} \sum_{j=1}^{N_n} e_{j,l} \lambda_{j,k}^* \right)^2 \right] \leq K/N_n.$$

Therefore,

$$\frac{1}{k_n} \sum_{l=1}^{k_n} \left( \frac{1}{N_n} \sum_{j=1}^{N_n} e_{j,l} \lambda_{j,k}^* \right)^2 = O_p(N_n^{-1}) = o_p(1). \quad (\text{SA.50})$$

The assertion of part (c) then follows from (SA.49) and (SA.50).

(d) Like (SA.49), we can derive

$$\frac{1}{k_n N_n} \left| \gamma^\top \mathcal{E}_n^{\prime\top} \Lambda_{n,\cdot,k}^* \right| \leq \left( \frac{1}{k_n} \sum_{l=1}^{k_n} \left( \frac{1}{N_n} \sum_{j=1}^{N_n} e'_{j,l} \lambda_{j,k}^* \right)^2 \right)^{1/2}. \quad (\text{SA.51})$$

We further note that,

$$\begin{aligned} \mathbb{E} \left[ \left( \frac{1}{N_n} \sum_{j=1}^{N_n} e'_{j,l} \lambda_{j,k}^* \right)^2 \right] &\leq \mathbb{E} \left[ \frac{1}{N_n} \sum_{j=1}^{N_n} (e'_{j,l} \lambda_{j,k}^*)^2 \right] \\ &\leq \frac{K}{N_n} \sum_{j=1}^{N_n} \mathbb{E} [(e'_{j,l})^2] \leq K \Delta_n, \end{aligned}$$

where the first inequality is by Jensen's inequality, the second inequality holds because  $\lambda_{j,k}^*$  is bounded and the last inequality can be derived similarly as (SA.47). In view of (SA.51), the assertion of part (d) readily follows.

(e) From the definitions of  $\Lambda_n^*$  and  $\tilde{\Lambda}_n^*$  respectively from (3.9) and (SA.40), we see that (recall  $q = \tau -$ )

$$\tilde{\Lambda}_n^* - \Lambda_n^* = \left( \left( \beta_{1,\tau}^* - \tilde{\beta}_{n,1,\tau} \right) \lambda_{Z,\tau-}, \dots, \left( \beta_{n,N_n,\tau}^* - \tilde{\beta}_{n,N_n,\tau} \right) \lambda_{Z,\tau-} \right)^\top.$$

Therefore, by Assumption 7,

$$\frac{1}{N_n} (\tilde{\Lambda}_n^* - \Lambda_n^*)^\top (\tilde{\Lambda}_n^* - \Lambda_n^*) = o_p(1). \quad (\text{SA.52})$$

That is,  $N_n^{-1} \|\tilde{\Lambda}_n^* - \Lambda_n^*\|^2 = o_p(1)$ . Since  $N_n^{-1} \Lambda_n^{*\top} \Lambda_n^* \xrightarrow{\mathbb{P}} M_\Lambda^*$  by Assumption 8, the estimate above readily implies the assertions in part (e). *Q.E.D.*

We are now ready to prove part (a) and part (b) of Theorem 2. We remind the reader that we fix  $q = \tau -$  for proving these parts.

PROOF OF THEOREM 2(a). Step 1. We prove part (a) of Theorem 2 in several steps. In this step, we show that

$$\sup_{\gamma \in \Gamma_n} |\Xi_n(\gamma) - \Xi_n^*(\gamma)| = o_p(1), \quad (\text{SA.53})$$

where  $\Xi_n(\cdot)$  and  $\Xi_n^*(\cdot)$  are defined as

$$\Xi_n(\gamma) \equiv \frac{1}{k_n^2 N_n} \gamma^\top \hat{X}_n^\top \hat{X}_n \gamma, \quad \Xi_n^*(\gamma) \equiv \frac{1}{k_n^2 N_n} \gamma^\top F_n \Lambda_n^{*\top} \Lambda_n^* F_n^\top \gamma. \quad (\text{SA.54})$$

Below, we denote the  $(j, l)$  element of  $\hat{X}_n$  by

$$\xi_{n,j,l} \equiv \frac{\Delta_{i(n,\tau-)+l}^n Y_j \wedge u_n \vee (-u_n) - \tilde{\beta}_{n,j,\tau} \Delta_{i(n,\tau-)+l}^n Z}{\sqrt{\Delta_n}}.$$

We set

$$\begin{aligned}\xi'_{n,j,l} &\equiv \Delta_n^{-1/2} \int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n} \alpha_{j,s} ds \\ &\quad + \Delta_n^{-1/2} \int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n} (\lambda_{j,s} - \tilde{\beta}_{n,j,\tau} \lambda_{Z,s})^\top df_s + \Delta_n^{-1/2} \Delta_n^n_{i(n,\tau-)+l} \epsilon_j.\end{aligned}$$

Note that

$$\mathbb{E} |\xi_{n,j,l} - \xi'_{n,j,l}|^2 \leq K \Delta_n. \quad (\text{SA.55})$$

We now define  $\hat{X}'_n$  as a  $N_n \times k_n$  matrix whose  $(j, l)$  element is given by  $\xi'_{n,j,l}$  and let

$$\Xi'_n(\gamma) = \frac{1}{k_n^2 N_n} \gamma^\top \hat{X}'_n{}^\top \hat{X}'_n \gamma.$$

By (SA.55),

$$\frac{1}{k_n N_n} \left\| \hat{X}_n - \hat{X}'_n \right\|^2 = \frac{1}{k_n N_n} \sum_{j=1}^{N_n} \sum_{l=1}^{k_n} |\xi_{n,j,l} - \xi'_{n,j,l}|^2 = o_p(1). \quad (\text{SA.56})$$

By the Cauchy–Schwarz inequality and the triangle inequality,

$$\begin{aligned}\sup_{\gamma \in \Gamma_n} |\Xi_n(\gamma) - \Xi'_n(\gamma)| &= \frac{1}{k_n^2 N_n} \sup_{\gamma \in \Gamma_n} \left| \gamma^\top \left( \hat{X}_n{}^\top \hat{X}_n - \hat{X}'_n{}^\top \hat{X}'_n \right) \gamma \right| \\ &\leq \frac{1}{k_n^2 N_n} \sup_{\gamma \in \Gamma_n} \|\gamma\|^2 \left\| \hat{X}_n{}^\top \hat{X}_n - \hat{X}'_n{}^\top \hat{X}'_n \right\| \\ &= \frac{1}{k_n N_n} \left\| \hat{X}_n{}^\top \hat{X}_n - \hat{X}'_n{}^\top \hat{X}'_n \right\| \\ &\leq \frac{2}{k_n N_n} \left\| \hat{X}'_n \right\| \left\| \hat{X}_n - \hat{X}'_n \right\| + \frac{1}{k_n N_n} \left\| \hat{X}_n - \hat{X}'_n \right\|^2.\end{aligned}$$

It is easy to see that  $\|\hat{X}'_n\| = O_p(\sqrt{k_n N_n})$ . Hence, by (SA.56),

$$\sup_{\gamma \in \Gamma_n} |\Xi_n(\gamma) - \Xi'_n(\gamma)| = o_p(1). \quad (\text{SA.57})$$

To show (SA.53), it remains to show that  $\sup_{\gamma \in \Gamma_n} |\Xi'_n(\gamma) - \Xi_n^*(\gamma)| = o_p(1)$ . We note that, by a standard result for spot covariance estimation

$$F_n{}^\top F_n / k_n \xrightarrow{\mathbb{P}} \Sigma_f. \quad (\text{SA.58})$$

In particular,  $\|F_n\| = O_p(k_n^{1/2})$ . Hence,

$$\sup_{\gamma \in \Gamma_n} \left\| \gamma^\top F_n / k_n \right\| \leq \sup_{\gamma \in \Gamma_n} \|\gamma\| \|F_n\| / k_n = O_p(1). \quad (\text{SA.59})$$

Under Assumption 8,  $\Lambda_n^{*\top} \Lambda_n^* = O_p(N_n)$ . It then follows that

$$\sup_{\gamma \in \Gamma_n} \Xi_n^*(\gamma) = O_p(1). \quad (\text{SA.60})$$

Recall the definitions in (3.9), (SA.40) and (SA.41). We can decompose  $\hat{X}'_n$  as

$$\hat{X}'_n = \tilde{\Lambda}_n^* F_n^\top + \mathcal{E}_n + \mathcal{E}'_n. \quad (\text{SA.61})$$

Hence,

$$\hat{X}'_n - \Lambda_n^* F_n^\top = (\tilde{\Lambda}_n^* - \Lambda_n^*) F_n^\top + \mathcal{E}_n + \mathcal{E}'_n. \quad (\text{SA.62})$$

We can then decompose

$$\begin{aligned} \Xi'_n(\gamma) - \Xi_n^*(\gamma) &= \frac{2}{k_n^2 N_n} \gamma^\top F_n \Lambda_n^{*\top} \left[ (\tilde{\Lambda}_n^* - \Lambda_n^*) F_n^\top + \mathcal{E}_n + \mathcal{E}'_n \right] \gamma \\ &\quad + \frac{1}{k_n^2 N_n} \gamma^\top \left( (\tilde{\Lambda}_n^* - \Lambda_n^*) F_n^\top + \mathcal{E}_n + \mathcal{E}'_n \right)^\top \left( (\tilde{\Lambda}_n^* - \Lambda_n^*) F_n^\top + \mathcal{E}_n + \mathcal{E}'_n \right) \gamma. \end{aligned} \quad (\text{SA.63})$$

By Lemma SA4(a,b),

$$\frac{1}{k_n^2 N_n} \sup_{\gamma \in \Gamma_n} \gamma^\top \mathcal{E}_n^\top \mathcal{E}_n \gamma = o_p(1), \quad \frac{1}{k_n^2 N_n} \sup_{\gamma \in \Gamma_n} \gamma^\top \mathcal{E}'_n{}^\top \mathcal{E}'_n \gamma = o_p(1). \quad (\text{SA.64})$$

Further using the Cauchy–Schwarz inequality, we can deduce that  $\sup_{\gamma \in \Gamma_n} |\gamma^\top \mathcal{E}_n^\top \mathcal{E}'_n \gamma| = o_p(1)$ ; hence,

$$\frac{1}{k_n^2 N_n} \sup_{\gamma \in \Gamma_n} \gamma^\top (\mathcal{E}_n + \mathcal{E}'_n)^\top (\mathcal{E}_n + \mathcal{E}'_n) \gamma = o_p(1). \quad (\text{SA.65})$$

In addition, by Lemma SA4(e) and (SA.59)

$$\sup_{\gamma \in \Gamma_n} \frac{1}{k_n^2 N_n} \gamma^\top F_n^\top (\tilde{\Lambda}_n^* - \Lambda_n^*)^\top (\tilde{\Lambda}_n^* - \Lambda_n^*) F_n^\top \gamma = o_p(1). \quad (\text{SA.66})$$

By (SA.65) and (SA.66), as well as the Cauchy–Schwarz inequality, we deduce

$$\frac{1}{k_n^2 N_n} \sup_{\gamma \in \Gamma_n} \gamma^\top \left( (\tilde{\Lambda}_n^* - \Lambda_n^*) F_n^\top + \mathcal{E}_n + \mathcal{E}'_n \right)^\top \left( (\tilde{\Lambda}_n^* - \Lambda_n^*) F_n^\top + \mathcal{E}_n + \mathcal{E}'_n \right) \gamma = o_p(1). \quad (\text{SA.67})$$

By (SA.60) and the Cauchy–Schwarz inequality, (SA.67) further implies that

$$\frac{2}{k_n^2 N_n} \sup_{\gamma \in \Gamma_n} \left| \gamma^\top F_n \Lambda_n^{*\top} \left[ (\tilde{\Lambda}_n^* - \Lambda_n^*) F_n^\top + \mathcal{E}_n + \mathcal{E}'_n \right] \gamma \right| = o_p(1). \quad (\text{SA.68})$$

By (SA.63), (SA.67) and (SA.68), we deduce  $\sup_{\gamma \in \Gamma_n} |\Xi'_n(\gamma) - \Xi_n^*(\gamma)| = o_p(1)$  and, hence, (SA.53) as wanted.

Step 2. In this step, we show that

$$S_n^* (\hat{F}_n^{*\top} F_n / k_n) \Sigma_f^{-1/2} H \xrightarrow{\mathbb{P}} I_r, \quad (\text{SA.69})$$

where we recall that  $S_n^* = \text{diag}(\text{sign}(\hat{F}_n^{*\top} F_n (F_n^\top F_n / k_n)^{-1/2} H))$  and  $H$  is the ordered eigenvector matrix of  $M_C^*$ . Below, we denote by  $D_j$  the  $j$ th largest eigenvalue of  $M_C^*$  and write  $D = \text{diag}(D_1, \dots, D_r)$ .



We first show that

$$\sup_{\gamma \in \Gamma_n} \Xi_n^*(\gamma) \xrightarrow{\mathbb{P}} D_1. \quad (\text{SA.70})$$

To see this, we note that we can represent  $\gamma \in \Gamma_n$  as

$$\gamma = F_n(F_n^\top F_n/k_n)^{-1/2} H\delta + \tilde{\gamma}, \quad (\text{SA.71})$$

where  $\tilde{\gamma}$  is the projection error of  $\gamma$  onto the column space of  $F_n$  such that  $F_n^\top \tilde{\gamma} = 0$ . We can then rewrite

$$\begin{aligned} \sup_{\gamma \in \Gamma_n} \Xi_n^*(\gamma) &= \sup_{\|\delta\| \leq 1} \delta^\top H^\top M_{C,n}^* H \delta, \quad \text{where} \\ M_{C,n}^* &\equiv \left( \frac{F_n^\top F_n}{k_n} \right)^{1/2} \left( \frac{\Lambda_n^{*\top} \Lambda_n^*}{N_n} \right) \left( \frac{F_n^\top F_n}{k_n} \right)^{1/2}. \end{aligned}$$

Hence,  $\sup_{\gamma \in \Gamma_n} \Xi_n^*(\gamma)$  is the largest eigenvalue of  $M_{C,n}^*$ . By (SA.58) and Assumption 8,

$$M_{C,n}^* \xrightarrow{\mathbb{P}} \Sigma_f^{1/2} M_\Lambda^* \Sigma_f^{1/2} \equiv M_C^*.$$

Since the mapping for calculating the unique largest eigenvalue is continuous, we deduce (SA.70) by using the continuous mapping theorem.

By the construction of  $\hat{F}_n$ , its first column  $\hat{F}_{n,1}$  satisfies

$$\Xi_n(\hat{F}_{n,1}) = \sup_{\gamma \in \Gamma_n} \Xi_n(\gamma).$$

By (SA.53),  $\sup_{\gamma \in \Gamma_n} \Xi_n(\gamma) = \sup_{\gamma \in \Gamma_n} \Xi_n^*(\gamma) + o_p(1)$ , which implies  $\Xi_n(\hat{F}_{n,1}) \xrightarrow{\mathbb{P}} D_1$  because of (SA.70). Using the uniform convergence result in (SA.53), we further deduce

$$\Xi_n^*(\hat{F}_{n,1}) \xrightarrow{\mathbb{P}} D_1. \quad (\text{SA.72})$$

We now represent  $\hat{F}_{n,1}$  in the format of (SA.71), that is,

$$\hat{F}_{n,1} = F_n(F_n^\top F_n/k_n)^{-1/2} H\hat{\delta}_1 + \tilde{\gamma}_1, \quad (\text{SA.73})$$

such that  $F_n^\top \tilde{\gamma}_1 = 0$ . From (SA.72) and (SA.73), we see

$$\begin{aligned} o_p(1) &= \Xi_n^*(\hat{F}_{n,1}) - D_1 \\ &= \hat{\delta}_1^\top H^\top M_{C,n}^* H \hat{\delta}_1 - D_1 \\ &= \hat{\delta}_1^\top H^\top (M_{C,n}^* - M_C^*) H \hat{\delta}_1 + \hat{\delta}_1^\top H^\top M_C^* H \hat{\delta}_1 - D_1 \\ &= \hat{\delta}_1^\top H^\top (M_{C,n}^* - M_C^*) H \hat{\delta}_1 + \hat{\delta}_1^\top D \hat{\delta}_1 - D_1, \end{aligned}$$

where the last line follows from the eigenvalue decomposition  $M_C^* = H D H^\top$ . Since  $\|\hat{\delta}_1\| \leq 1$  and  $M_{C,n}^* - M_C^* = o_p(1)$ , the above display implies that

$$\hat{\delta}_1^\top D \hat{\delta}_1 - D_1 = o_p(1).$$

Since  $D_1$  is the unique largest eigenvalue, this further implies that  $\hat{\delta}_{11}^2 \xrightarrow{\mathbb{P}} 1$  and  $\hat{\delta}_{1j}^2 \xrightarrow{\mathbb{P}} 0$  for  $j \geq 2$ . In particular,  $\|\hat{\delta}_1\| \xrightarrow{\mathbb{P}} 1$  which implies that  $\tilde{\gamma}_1^\top \tilde{\gamma}_1 / k_n \xrightarrow{\mathbb{P}} 0$ .

Let  $S_{n,j}^*$  denote the  $j$ th diagonal element of  $S_n^*$ . Note that by (SA.73),

$$\hat{F}_{n,1}^\top F_n / k_n = \hat{\delta}_1^\top H^\top (F_n^\top F_n / k_n)^{1/2}.$$

Hence,

$$\hat{\delta}_1^\top = (\hat{F}_{n,1}^\top F_n / k_n) (F_n^\top F_n / k_n)^{-1/2} H.$$

By the definition of  $S_{n,1}^*$ , the first element of  $S_{n,1}^* (\hat{F}_{n,1}^\top F_n / k_n) (F_n^\top F_n / k_n)^{-1/2} H$  is nonnegative.

Hence,

$$\begin{aligned} & S_{n,1}^* (\hat{F}_{n,1}^\top F_n / k_n) (F_n^\top F_n / k_n)^{-1/2} H \\ &= \left( |\hat{\delta}_{11}|, S_{n,1}^* \hat{\delta}_{12}, \dots, S_{n,1}^* \hat{\delta}_{1r} \right) \xrightarrow{\mathbb{P}} (1, 0, \dots, 0). \end{aligned}$$

By (SA.58), we further deduce that

$$S_{n,1}^* (\hat{F}_{n,1}^\top F_n / k_n) \Sigma_f^{-1/2} H \xrightarrow{\mathbb{P}} (1, 0, \dots, 0),$$

which shows the convergence in (SA.69) for the first row.

By repeating the same argument (by setting  $\Gamma_n$  as the subspace orthogonal to previous eigenvectors), we can prove the convergence in (SA.69) for the  $j$ th row,  $2 \leq j \leq r$ .

Step 3. In this step, we finish the proof for part (a) of Theorem 2. We denote

$$\tilde{D}_n = N_n^{-1} (\hat{\Lambda}_n^* - \Lambda_n^* \Sigma_f^{1/2} H S_n^*)^\top (\hat{\Lambda}_n^* - \Lambda_n^* \Sigma_f^{1/2} H S_n^*).$$

The assertion of part (a) can be rewritten as  $\text{Trace}[\tilde{D}_n] = o_p(1)$ .

We decompose

$$\tilde{D}_n = \tilde{D}_{n,1} - \tilde{D}_{n,2} - \tilde{D}_{n,2}^\top + \tilde{D}_{n,3},$$

where

$$\begin{cases} \tilde{D}_{n,1} \equiv N_n^{-1} \hat{\Lambda}_n^{*\top} \hat{\Lambda}_n^*, & \tilde{D}_{n,2} \equiv N_n^{-1} \hat{\Lambda}_n^{*\top} \Lambda_n^* \Sigma_f^{1/2} H S_n^*, \\ \tilde{D}_{n,3} \equiv N_n^{-1} S_n^* H^\top \Sigma_f^{1/2} \Lambda_n^{*\top} \Lambda_n^* \Sigma_f^{1/2} H S_n^*. \end{cases}$$

To prove  $\text{Trace}[\tilde{D}_n] = o_p(1)$ , it suffices to show that

$$\tilde{D}_{n,k} \xrightarrow{\mathbb{P}} D, \quad k = 1, 2, 3, \tag{SA.74}$$

where we recall that  $D$  is the diagonal matrix that collects the ordered eigenvalues of  $M_C^*$ . Below, we prove (SA.74) for each case.

*Case  $k = 1$ :* Recall that we partition  $\hat{F}_n = [\hat{F}_n^* : \hat{F}_n^0]$ , where  $\hat{F}_n^*$  collects the first  $r$  columns of  $\hat{F}_n$ . We set

$$\hat{\Lambda}_n'^* = \frac{1}{k_n} \hat{X}_n' \hat{F}_n^* = \frac{1}{k_n} \left( \tilde{\Lambda}_n^* F_n^\top + \mathcal{E}_n + \mathcal{E}_n' \right) \hat{F}_n^*. \tag{SA.75}$$

Note that

$$\begin{aligned} \left\| \tilde{D}_{n,1} - N_n^{-1} \hat{\Lambda}_n'^*{}^\top \hat{\Lambda}_n'^* \right\| &= N_n^{-1} \left\| \hat{\Lambda}_n'^*{}^\top \hat{\Lambda}_n'^* - \hat{\Lambda}_n'^*{}^\top \hat{\Lambda}_n'^* \right\| \\ &= k_n^{-2} N_n^{-1} \left\| \hat{F}_n'^*{}^\top \hat{X}_n'^\top \hat{X}_n \hat{F}_n'^* - \hat{F}_n'^*{}^\top \hat{X}_n'^\top \hat{X}_n' \hat{F}_n'^* \right\| = o_p(1), \end{aligned} \quad (\text{SA.76})$$

where the first two equalities are by definition and the last one is by (SA.57). Subsequently, by (SA.75), we can decompose  $\tilde{D}_{n,1}$  as

$$\begin{aligned} \tilde{D}_{n,1} &= \frac{1}{k_n^2 N_n} \hat{F}_n'^*{}^\top \left( \tilde{\Lambda}_n'^*{}^\top F_n^\top + \mathcal{E}_n + \mathcal{E}'_n \right)^\top \left( \tilde{\Lambda}_n'^*{}^\top F_n^\top + \mathcal{E}_n + \mathcal{E}'_n \right) \hat{F}_n'^* + o_p(1) \\ &= \tilde{D}_{n,1,1} + \tilde{D}_{n,1,2} + \tilde{D}_{n,1,2}^\top + \tilde{D}_{n,1,3} + o_p(1), \end{aligned}$$

where

$$\begin{cases} \tilde{D}_{n,1,1} \equiv \left( \hat{F}_n'^*{}^\top F_n / k_n \right) \left( \tilde{\Lambda}_n'^*{}^\top \tilde{\Lambda}_n'^* / N_n \right) \left( F_n^\top \hat{F}_n'^* / k_n \right), \\ \tilde{D}_{n,1,2} \equiv \left( k_n^{-1} N_n^{-1} \hat{F}_n'^*{}^\top \left( \mathcal{E}_n + \mathcal{E}'_n \right)^\top \tilde{\Lambda}_n'^* \right) \left( F_n^\top \hat{F}_n'^* / k_n \right), \\ \tilde{D}_{n,1,3} \equiv k_n^{-2} N_n^{-1} \hat{F}_n'^*{}^\top \left( \mathcal{E}_n + \mathcal{E}'_n \right)^\top \left( \mathcal{E}_n + \mathcal{E}'_n \right) \hat{F}_n'^*. \end{cases}$$

From (SA.69),

$$\frac{1}{k_n} \hat{F}_n'^*{}^\top F_n - S_n^* H^\top \Sigma_f^{1/2} = o_p(1), \quad \frac{1}{k_n} \hat{F}_n'^*{}^\top F_n = O_p(1). \quad (\text{SA.77})$$

Hence, recalling that  $H$  is the eigenvector matrix of  $M_C^* = \Sigma_f^{1/2} M_\Lambda^* \Sigma_f^{1/2}$  and  $S_n^*$  is a diagonal matrix with  $\pm 1$  on its diagonal, we deduce

$$\tilde{D}_{n,1,1} = S_n^* H^\top \Sigma_f^{1/2} M_\Lambda^* \Sigma_f^{1/2} H S_n^* + o_p(1) = D + o_p(1).$$

By Lemma SA4, we see that  $\tilde{D}_{n,1,2}$  and  $\tilde{D}_{n,1,3}$  are both  $o_p(1)$ . From these estimates, (SA.74) for the case  $k = 1$  readily follows.

*Case  $k = 2$ :* By (SA.76) and the Cauchy–Schwarz inequality,

$$\tilde{D}_{n,2} \equiv N_n^{-1} \hat{\Lambda}_n'^*{}^\top \Lambda_n^* \Sigma_f^{1/2} H S_n^* + o_p(1).$$

By (SA.75), we can thus decompose  $\tilde{D}_{n,2}$  as  $\tilde{D}_{n,2} = \tilde{D}_{n,2,1} + \tilde{D}_{n,2,2} + o_p(1)$  where

$$\begin{cases} \tilde{D}_{n,2,1} \equiv \left( \hat{F}_n'^*{}^\top F_n / k_n \right) \left( \tilde{\Lambda}_n'^*{}^\top \Lambda_n^* / N_n \right) \Sigma_f^{1/2} H S_n^*, \\ \tilde{D}_{n,2,2} \equiv \left( k_n^{-1} N_n^{-1} \hat{F}_n'^*{}^\top \left( \mathcal{E}_n + \mathcal{E}'_n \right)^\top \Lambda_n^* \right) \Sigma_f^{1/2} H S_n^*. \end{cases}$$

By (SA.77) and Lemma SA4(e), we deduce

$$\tilde{D}_{n,2,1} = S_n^* H^\top \Sigma_f^{1/2} M_\Lambda^* \Sigma_f^{1/2} H S_n^* + o_p(1) = D + o_p(1).$$

By Lemma SA4(c,d),  $\tilde{D}_{n,2,2} = o_p(1)$ . This proves (SA.74) for the case  $k = 2$ .

*Case  $k = 3$ :* By Assumption 8, it is obvious that

$$\tilde{D}_{n,3} = S_n^* H^\top \Sigma_f^{1/2} M_\Lambda^* \Sigma_f^{1/2} H S_n^* + o_p(1) = D + o_p(1).$$

This finishes the proof of (SA.74) and, hence, part (a) of Theorem 2. *Q.E.D.*

PROOF OF THEOREM 2(b). We fix  $j \in \{r+1, \dots, \bar{r}\}$ . Recall that  $\hat{\Lambda}_{n,j}$  denote the  $j$ th column of  $\hat{\Lambda}_n$ . By the definitions of  $\hat{\Lambda}_n$  and  $\hat{F}_n$ ,

$$\frac{1}{N_n} \hat{\Lambda}_{n,j}^\top \hat{\Lambda}_{n,j} = \Xi_n(\hat{F}_{n,j}). \quad (\text{SA.78})$$

Like in (SA.73), for each  $k \in \{1, \dots, r\}$ , we can represent

$$\hat{F}_{n,k} = F_n(F_n^\top F_n/k_n)^{-1/2} H \hat{\delta}_k + \tilde{\gamma}_k, \quad (\text{SA.79})$$

where  $F_n^\top \tilde{\gamma}_k = 0$ . Following a similar argument as in Step 2 of the proof of Theorem 2(a), we can show that, for each  $k, k' \in \{1, \dots, r\}$  with  $k \neq k'$ ,

$$\hat{\delta}_{kk}^2 \xrightarrow{\mathbb{P}} 1, \quad \hat{\delta}_{kk'} \xrightarrow{\mathbb{P}} 0, \quad \tilde{\gamma}_k^\top \tilde{\gamma}_{k'}/k_n \xrightarrow{\mathbb{P}} 0. \quad (\text{SA.80})$$

We also represent

$$\hat{F}_{n,j} = F_n(F_n^\top F_n/k_n)^{-1/2} H \hat{\delta}_j + \tilde{\gamma}_j, \quad (\text{SA.81})$$

where  $F_n^\top \tilde{\gamma}_j = 0$ . Since  $\hat{F}_{n,j}^\top \hat{F}_{n,k}/k_n = 0$  for  $1 \leq k \leq r$  (because  $\hat{F}_n$  collects the eigenvectors of  $\hat{X}_n^\top \hat{X}_n$ ), we have

$$\hat{\delta}_j^\top \hat{\delta}_k + \tilde{\gamma}_j^\top \tilde{\gamma}_k/k_n = 0. \quad (\text{SA.82})$$

Since  $\tilde{\gamma}_k^\top \tilde{\gamma}_k/k_n \xrightarrow{\mathbb{P}} 0$  and  $\tilde{\gamma}_j^\top \tilde{\gamma}_j/k_n \leq 1$ , we have  $\tilde{\gamma}_j^\top \tilde{\gamma}_k/k_n = o_p(1)$  by the Cauchy–Schwarz inequality. Therefore,  $\hat{\delta}_j^\top \hat{\delta}_k = o_p(1)$  for  $1 \leq k \leq r$ . By (SA.80) above, this implies  $\hat{\delta}_j = o_p(1)$ . Hence,

$$\Xi_n^*(\hat{F}_{n,j}) = \hat{\delta}_j^\top H M_{C,n}^* H^\top \hat{\delta}_j = o_p(1). \quad (\text{SA.83})$$

By (SA.53),  $\Xi_n(\hat{F}_{n,j}) = o_p(1)$ . The assertion of part (b) readily follows from (SA.78). *Q.E.D.*

PROOF OF THEOREM 2(c). By Assumption 8,

$$H_p^\top \Sigma_{f,p}^{1/2} \frac{\Lambda_n^*(p)^\top \Lambda_n^*(q)}{N_n} \Sigma_{f,q}^{1/2} H_q \xrightarrow{\mathbb{P}} H_p^\top M_C^*(p, q) H_q. \quad (\text{SA.84})$$

We observe

$$\begin{aligned} & \frac{1}{N_n} \left\| \hat{\Lambda}_n^*(p)^\top \hat{\Lambda}_n^*(q) - S_n^*(p) H_p^\top \Sigma_{f,p}^{1/2} \Lambda_n^*(p)^\top \Lambda_n^*(q) \Sigma_{f,q}^{1/2} H_q S_n^*(q) \right\| \\ & \leq \frac{1}{N_n} \left\| \left( \hat{\Lambda}_n^*(p) - \Lambda_n^*(p) \Sigma_{f,p}^{1/2} H_p S_n^*(p) \right)^\top \Lambda_n^*(q) \Sigma_{f,q}^{1/2} H_q \right\| \\ & \quad + \frac{1}{N_n} \left\| H_p^\top \Sigma_{f,p}^{1/2} \Lambda_n^*(p)^\top \left( \hat{\Lambda}_n^*(q) - \Lambda_n^*(q) \Sigma_{f,q}^{1/2} H_q S_n^*(q) \right) \right\| \\ & \quad + \frac{1}{N_n} \left\| \left( \hat{\Lambda}_n^*(p) - \Lambda_n^*(p) \Sigma_{f,p}^{1/2} H_p S_n^*(p) \right)^\top \left( \hat{\Lambda}_n^*(q) - \Lambda_n^*(q) \Sigma_{f,q}^{1/2} H_q S_n^*(q) \right) \right\|. \end{aligned}$$

By the Cauchy–Schwarz inequality and Theorem 2(a), we deduce that the terms on the majorant side of the above display are all  $o_p(1)$ . Hence, by (SA.84),

$$\frac{1}{N_n} \hat{\Lambda}_n^*(p)^\top \hat{\Lambda}_n^*(q) - S_n^*(p) H_p^\top M_C^*(p, q) H_q S_n^*(q) = o_p(1). \quad (\text{SA.85})$$

In particular,

$$\frac{1}{N_n} \hat{\Lambda}_n^*(p)^\top \hat{\Lambda}_n^*(q) = O_p(1). \quad (\text{SA.86})$$

By Theorem 2(b),

$$\frac{1}{N_n} \hat{\Lambda}_n^0(p)^\top \hat{\Lambda}_n^0(q) = o_p(1). \quad (\text{SA.87})$$

By the Cauchy–Schwarz inequality, (SA.86) and (SA.87), we deduce

$$\frac{1}{N_n} \hat{\Lambda}_n^*(p)^\top \hat{\Lambda}_n^0(q) = o_p(1). \quad (\text{SA.88})$$

The assertion of part (c) then follows from (SA.85), (SA.87) and (SA.88). *Q.E.D.*

PROOF OF THEOREM 2(d). By part (c) of Theorem 2,

$$\begin{aligned} \text{Trace}[\widehat{M}_{C,n}(q, q)] &= \text{Trace}[S_n^*(q) H_q^\top M_C^*(q, q) H_q S_n^*(q)] + o_p(1) \\ &= \text{Trace}[M_C^*(q, q)] + o_p(1) \\ &= \text{Trace}[M_\Lambda^*(q, q) \Sigma_{f,q}] + o_p(1), \end{aligned}$$

where the second inequality follows from the orthogonality of  $H_q S_n^*(q)$  and the last line holds because  $M_C^*(q, q) = \Sigma_{f,q}^{1/2} M_\Lambda^*(q, q) \Sigma_{f,q}^{1/2}$ . We also note from (SA.56) that

$$\frac{1}{k_n N_n} \|\hat{X}_n(q)\|^2 = \frac{1}{k_n N_n} \|\hat{X}'_n(q)\|^2 + o_p(1).$$

Hence, it remains to show that

$$\frac{1}{k_n N_n} \|\hat{X}'_n(q)\|^2 \xrightarrow{\mathbb{P}} \text{Trace}[M_\Lambda^*(q, q) \Sigma_{f,q}] + M_\epsilon(q). \quad (\text{SA.89})$$

To show (SA.89), we consider the following decomposition:

$$\begin{aligned} \|\hat{X}'_n(q)\|^2 &= \text{Trace} \left[ \hat{X}'_n(q)^\top \hat{X}'_n(q) \right] \\ &= \text{Trace} \left[ \tilde{\Lambda}_n^*(q)^\top \tilde{\Lambda}_n^*(q) F_n(q)^\top F_n(q) \right] \\ &\quad + \text{Trace} \left[ (\mathcal{E}_n(q) + \mathcal{E}'_n(q))^\top (\mathcal{E}_n(q) + \mathcal{E}'_n(q)) \right] \\ &\quad + 2 \text{Trace} \left[ F_n(q) \tilde{\Lambda}_n^*(q)^\top (\mathcal{E}_n(q) + \mathcal{E}'_n(q)) \right]. \end{aligned} \quad (\text{SA.90})$$

By Lemma SA4(e) and (SA.58),

$$\frac{1}{k_n N_n} \text{Trace} \left[ \tilde{\Lambda}_n^*(q)^\top \tilde{\Lambda}_n^*(q) F_n(q)^\top F_n(q) \right] \xrightarrow{\mathbb{P}} \text{Trace} [M_\Lambda^*(q, q) \Sigma_{f,q}]. \quad (\text{SA.91})$$

In the proof of Lemma SA4(c,d), we have shown that

$$\frac{1}{k_n} \left\| \frac{1}{N_n} (\mathcal{E}_n(q) + \mathcal{E}'_n(q))^\top \Lambda_n^*(q) \right\|^2 = o_p(1).$$

In addition, by (SA.52),

$$\begin{aligned} & \frac{1}{k_n} \left\| \frac{1}{N_n} (\mathcal{E}_n(q) + \mathcal{E}'_n(q))^\top (\tilde{\Lambda}_n^*(q) - \Lambda_n^*(q)) \right\|^2 \\ & \leq \frac{\|\mathcal{E}_n(q) + \mathcal{E}'_n(q)\|^2}{k_n N_n} \cdot \frac{\|\tilde{\Lambda}_n^*(q) - \Lambda_n^*(q)\|^2}{N_n} = o_p(1). \end{aligned}$$

Hence,  $\|(\mathcal{E}_n(q) + \mathcal{E}'_n(q))^\top \tilde{\Lambda}_n^*(q)\| = o_p(N_n k_n^{1/2})$ . Also note that  $\|F_n(q)\| = O_p(k_n^{1/2})$ . Therefore, by the Cauchy–Schwarz inequality,

$$\left\| \frac{1}{k_n N_n} (\mathcal{E}_n(q) + \mathcal{E}'_n(q))^\top \tilde{\Lambda}_n^*(q) F_n(q)^\top \right\| \leq \frac{1}{k_n N_n} \|F_n(q)\| \left\| (\mathcal{E}_n(q) + \mathcal{E}'_n(q))^\top \tilde{\Lambda}_n^*(q) \right\| = o_p(1).$$

Consequently,

$$\frac{1}{k_n N_n} \text{Trace} \left[ F_n(q) \tilde{\Lambda}_n^*(q)^\top (\mathcal{E}_n(q) + \mathcal{E}'_n(q)) \right] = o_p(1). \quad (\text{SA.92})$$

In view of (SA.90), (SA.91) and (SA.92), (SA.89) will be implied by

$$\frac{1}{k_n N_n} \text{Trace} \left[ (\mathcal{E}_n(q) + \mathcal{E}'_n(q))^\top (\mathcal{E}_n(q) + \mathcal{E}'_n(q)) \right] \xrightarrow{\mathbb{P}} M_\epsilon(q). \quad (\text{SA.93})$$

Finally, we show (SA.93). For each  $j$ , we denote

$$\begin{aligned} \xi_{n,j} & \equiv \frac{1}{k_n} \sum_{l=1}^{k_n} \left( \frac{\Delta_{i(n,q)+l}^n \epsilon_j}{\sqrt{\Delta_n}} \right)^2, \\ \xi'_{n,j} & \equiv \frac{1}{k_n \Delta_n} \int_{i(n,q)\Delta_n}^{i(n,q)\Delta_n + k_n \Delta_n} \tilde{\sigma}_{j,u}^2 du, \quad \xi''_{n,j} \equiv \xi_{n,j} - \xi'_{n,j}. \end{aligned}$$

Then, we can decompose

$$\begin{aligned} \frac{1}{k_n N_n} \text{Trace} \left[ \mathcal{E}_n(q)^\top \mathcal{E}_n(q) \right] & = \frac{1}{k_n N_n} \sum_{j=1}^{N_n} \sum_{l=1}^{k_n} \left( \frac{\Delta_{i(n,q)+l}^n \epsilon_j}{\sqrt{\Delta_n}} \right)^2 \\ & = \frac{1}{N_n} \sum_{j=1}^{N_n} \xi'_{n,j} + \frac{1}{N_n} \sum_{j=1}^{N_n} \xi''_{n,j}. \end{aligned}$$

We note that conditional on  $\mathcal{F}_{i(n,q)\Delta_n}$ , the variables  $(\xi''_{n,j})_{1 \leq j \leq N_n}$  are uncorrelated with zero mean and bounded variances. Hence,

$$\frac{1}{N_n} \sum_{j=1}^{N_n} \xi''_{n,j} = o_p(1). \quad (\text{SA.94})$$

In addition, we note that

$$\begin{aligned} \frac{1}{N_n} \sum_{j=1}^{N_n} \xi'_{n,j} - \frac{1}{N_n} \sum_{j=1}^{N_n} \tilde{\sigma}_{j,q}^2 & = \frac{1}{N_n} \sum_{j=1}^{N_n} \frac{1}{k_n \Delta_n} \int_{i(n,q)\Delta_n}^{i(n,q)\Delta_n + k_n \Delta_n} (\tilde{\sigma}_{j,u}^2 - \tilde{\sigma}_{j,q}^2) du \\ & = O_p(k_n^{1/2} \Delta_n^{1/2}) = o_p(1). \end{aligned}$$

It readily follows that

$$\frac{1}{N_n} \sum_{j=1}^{N_n} \xi'_{n,j} \xrightarrow{\mathbb{P}} M_\epsilon(q). \quad (\text{SA.95})$$

By (SA.94) and (SA.95),

$$\frac{1}{k_n N_n} \text{Trace} \left[ \mathcal{E}_n(q)^\top \mathcal{E}_n(q) \right] \xrightarrow{\mathbb{P}} M_\epsilon(q). \quad (\text{SA.96})$$

We further note that

$$\frac{1}{k_n N_n} \text{Trace} \left[ \mathcal{E}'_n(q)^\top \mathcal{E}'_n(q) \right] = \frac{1}{k_n N_n} \sum_{j=1}^{N_n} \sum_{l=1}^{k_n} (e'_{j,l})^2 = O_p(\Delta_n). \quad (\text{SA.97})$$

With an appeal to the Cauchy–Schwarz inequality, we deduce (SA.93) from (SA.96) and (SA.97).

This finishes the proof of part (d) of Theorem 2.

*Q.E.D.*

### SA.5 Proof of Theorem 3

(a) Firstly, by Theorem 2(c,d), it is obvious that  $\tilde{L}_n(\eta, \tau) = O_p(1)$ . Hence, the quantile  $cv_{n,\alpha} = O_p(1)$ . Next, we consider the case under the null hypothesis, so  $M_C^*(p, q)$  coincides with  $M_C(p, q)$ .

We partition  $\tilde{\zeta}_q^\top = (\tilde{\zeta}_q^{*\top}, \tilde{\zeta}_q^{0\top})$ , where  $\tilde{\zeta}_q^*$  is  $r$ -dimensional. By Theorem 2(c,d), we have, for  $s \in \{\eta, \tau\}$ ,

$$\begin{cases} \tilde{A}_n(s) = \sum_{p,q \in \{s-, s+\}} \tilde{w}_{n,p} \tilde{w}_{n,q} \tilde{\zeta}_p^{*\top} S_n^*(p) H_p^\top M_C(p, q) H_q S_n^*(q) \tilde{\zeta}_q^* + \sum_{q \in \{s-, s+\}} \tilde{w}_{n,q}^2 M_\epsilon(q) + o_p(1), \\ \tilde{B}_n(\eta, \tau) = \sum_{p \in \{\tau-, \tau+\}} \sum_{q \in \{\eta-, \eta+\}} \tilde{w}_{n,p} \tilde{w}_{n,q} \tilde{\zeta}_p^{*\top} S_n^*(p) H_p^\top M_C(p, q) H_q S_n^*(q) \tilde{\zeta}_q^* + o_p(1). \end{cases}$$

We note that the  $r$ -dimensional vectors  $H_q S_n^*(q) \tilde{\zeta}_q^*$  are, conditionally on  $\mathcal{F}$ , standard normal and mutually independent across  $q \in \{\tau-, \tau+, \eta-, \eta+\}$ . We also observe that for  $s \in \{\eta, \tau\}$ ,  $\Delta_{i(n,s)}^n Z \xrightarrow{\mathbb{P}} \Delta Z_s$ . Hence,

$$\left( H_q S_n^*(q) \tilde{\zeta}_q^*, \tilde{w}_{n,q} \right)_{q \in \{\tau-, \tau+, \eta-, \eta+\}} \xrightarrow{\mathcal{L}|\mathcal{F}} (\zeta_q, w_q)_{q \in \{\tau-, \tau+, \eta-, \eta+\}}, \quad (\text{SA.98})$$

where  $\xrightarrow{\mathcal{L}|\mathcal{F}}$  denotes the convergence of conditional law in probability. It follows that

$$\left( \tilde{A}_n(\eta), \tilde{A}_n(\tau), \tilde{B}_n(\eta, \tau) \right) \xrightarrow{\mathcal{L}|\mathcal{F}} (\mathcal{A}(\eta), \mathcal{A}(\tau), \mathcal{B}(\eta, \tau)).$$

Consequently,  $\tilde{L}_n(\eta, \tau) \xrightarrow{\mathcal{L}|\mathcal{F}} \mathcal{L}(\eta, \tau)$ . We further note that the  $\mathcal{F}$ -conditional distribution function of  $\mathcal{L}(\eta, \tau)$  is continuous and strictly increasing. Hence,  $cv_{n,\alpha} \xrightarrow{\mathbb{P}} cv_\alpha$ .

(b) The assertion on the asymptotic level follows from part (a) and Theorem 1. Under the alternative,  $\Delta_n^{-1} \hat{V}_n$  diverges to  $+\infty$  in probability by Proposition 1. The power property then follows from  $cv_{n,\alpha} = O_p(1)$ .

*Q.E.D.*

## References

JACOD, J., AND P. PROTTER (2012): *Discretization of Processes*. Springer.